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The automorphic membrane

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Abstract: We present a 1-loop toroidal membrane winding sum reproducing the conjectured M-theory, four-graviton, eight derivative, R^4 amplitude. The U-duality and toroidal membrane world-volume modular groups appear as a Howe dual pair in a larger, exceptional, group. A detailed analysis is carried out for M-theory compactified on a 3-torus, where the target-space $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ U-duality and $\mathrm{SL}(3,\mathbb{Z})$ world-volume modular groups are embedded in $E_{6(6)}(\mathbb{Z})$. Unlike previous semi-classical expansions, U-duality is built in manifestly and realized at the quantum level thanks to Fourier invariance of cubic characters. In addition to winding modes, a pair of new discrete, flux-like, quantum numbers are necessary to ensure invariance under the larger group. The action for these modes is of Born-Infeld type, interpolating between standard Polyakov and Nambu-Goto membrane actions. After integration over the membrane moduli, we recover the known R^4 amplitude, including membrane instantons. Divergences are disposed of by trading the non-compact volume integration for a compact integral over the two variables conjugate to the fluxes — a constant term computation in mathematical parlance. As byproducts, we suggest that, in line with membrane/fivebrane duality, the E_6 theta series also describes five-branes wrapped on T^6 in a manifestly U-duality invariant way. In addition we uncover a new action of E_6 on ten dimensional pure spinors, which may have implications for ten dimensional super Yang-Mills theory. An extensive review of SL(3) automorphic forms is included in an appendix.

Keywords: Solitons Monopoles and Instantons, Duality in Gauge Field Theories, M-Theory, String Duality.

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Contents

1. Introduction

Despite much evidence for its existence as a quantum theory, a tractable microscopic definition of M-theory is still missing years after the original conjecture [1, 2]. Various proposed definitions suffer either from a lack of computability or ties to specific backgrounds and energetics. Supermembranes remain one of the most promising candidates, because they (i) imply the equations of motion of eleven-dimensional supergravity as a consistency requirement (even at the classical level) [3], (ii) reduce to the ordinary type-II string by double reduction [4], and (iii) are equivalent to a continuous version of M(atrix) theory in light-cone gauge [5, 6, 7]. However, the nonlinearities of the membrane world-volume theory and the lack of an obvious genus expansion have so far stymied any attempt at direct quantization.

In a recent series of works, we have proposed to test the supermembrane M-theory hypothesis in a simple setting avoiding the usual quantization difficulties [8, 9]. Our proposal is that four graviton, eight derivative couplings in toroidally compactified M-theory follow from a *one-loop BPS membrane amplitude*. Here, "one-loop" refers to the membrane world-volume topology, namely a torus T^3, while "BPS" to the fact that only bosonic and fermionic zero-modes of the embedding coordinate and the world-volume metric contribute, fluctuations canceling by virtue of supersymmetry.

The basis for this proposal can be summarized as follows: Supersymmetry requires that the R^4 amplitude receives corrections from BPS states only, as exemplified by the one-loop computation of the R^4 amplitude in string theory, which indeed reduces to a sum of zero-mode worldsheet instanton configurations. In addition, the exact R^4 amplitude, determined on the basis of supersymmetry and U-duality [10]–[14], exhibits membrane instantons with only toroidal topology, wrapped on T^3 subtorii of the target space T^d.[1] Therefore, we expect that only toroidal membrane topologies T^3 contribute to this amplitude. Since U-duality relates membrane instantons to perturbative contributions, a treatment of membrane instantons that maintains *manifest U-duality* symmetry, will necessarily reproduce the complete R^4 amplitude, including its compactification-independent part.

In this Article, we demonstrate that our proposal [8], does yield the correct R^4 amplitude, in the simplest non-trivial case of M-theory compactified on a 3-torus. Instead of quantizing the classical membrane action,, we assume that the result of this quantization will reduce to a sum over discrete zero-mode configurations, invariant under all quantum symmetries. Invariance under the U-duality group $\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$ and the world-volume modular group $\mathrm{SL}(3,\mathbb{Z})$ is then shown to require a larger symmetry group, $E_{6(6)}(\mathbb{Z})$, under which the partition function is automorphic. In the process, we discover new discrete, flux-like, degrees of freedom on the membrane worldvolume, necessary to ensure U-duality

[1] Membrane instanton corrections in less supersymmetric settings have been discussed in [15].

invariance. We are also lead to a new prescription for dealing with the divergence associated to the membrane volume, which makes a crucial use of these new degrees of freedom. Our calculations are mathematically rigorous only in the degenerate sectors, corresponding to Kaluza-Klein contributions, the proof in the non-degenerate, membrane instanton sector falls short at a technical difficulty, for which we can only suggest a resolution.

This Article is organized as follows: In section 2, we outline the general strategy proposed in [8], and review the exact R^4 amplitude for M-theory compactified on a 3-torus, obtained in [12] on the basis of supersymmetry and U-duality. In section 3, we recall the construction of the E_6 theta series from our earlier work [9] and identify physical parameters $(\gamma_{AB}; g_{MN}, C_{MNP})$ with those entering the exceptional theta series. In section 4, we carry out the integration over the shape moduli of the world-volume T^3 and show how to produce the U-duality R^4 amplitude from a constant term integration which regulates the membrane world-volume integration. In section 5, we indicate how these results generalize to higher torii. We also point out the possible relevance of the E_6 theta series for five-branes compactified on T^6, which indicates that membrane/five-brane duality may be contained in our framework. Not unrelatedly, we also uncover a new action of E_6 on pure spinors in ten dimensions (first announced in [16]). It would be interesting to understand the implications for ten dimensional super Yang-Mills theory. Our conclusions, including various caveats and speculations are presented in section 6. Finally, appendix A gives more details on the E_6 theta series, appendix B describes relevant aspects of SL(3) automorphic forms which may usefully supplement our recent review [16], and appendix C is a sample membrane volume integral computation, illustrating the typical divergence of such integrals.

2. R^4 couplings, membranes and theta series: a review

In this section, we outline the general approach, first proposed in [8]. We then review the existing knowledge about R^4 couplings in 8 dimensions, which we shall attempt to derive from the membrane prospective.

2.1 R^4 couplings and theta correspondences

To summarize the salient steps of our construction, recall first that the analogous one-loop computation in type-II string theory compactified on T^d yields a partition function for string winding modes invariant under the world-volume modular group $\mathrm{SL}(2,\mathbb{Z})$ times the T-duality group $\mathrm{SO}(d,d,\mathbb{Z})$. The R^4 amplitude is then obtained by integrating over the fundamental domain of the toroidal lattice moduli space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$

$$f_{R^4}^{\text{1-loop}} = \int_{\mathrm{SO}(2)\backslash^{\mathrm{SL}(2,\mathbb{R})}/_{\mathrm{SL}(2,\mathbb{Z})}} d\gamma\, Z_{\mathrm{str}}(\gamma_{\alpha\beta}; g_{\mu\nu}, B_{\mu\nu})\,. \tag{2.1}$$

Here $\gamma_{\alpha\beta}$ is the unit volume constant metric on the world-volume 2-torus while $g_{\mu\nu}$ and $B_{\mu\nu}$ are the constant target space d-torus metric and 2-form gauge field. Correspondingly, the one-loop BPS membrane amplitude for M-theory compactified on T^d is a partition function for membrane winding modes, invariant under the modular group of the 3-torus $\mathrm{SL}(3,\mathbb{Z})$ times the U-duality group $E_d(\mathbb{Z})$. The M-theory four graviton, eight-derivative amplitude ought emerge after integrating over the fundamental domain of the moduli space of constant metrics on the toroidal world-volume:

$$f_{R^4}^{\text{exact}} \stackrel{?}{=} \int_{\mathrm{SO}(3)\backslash^{\mathrm{GL}(3,\mathbb{R})}/_{\mathrm{SL}(3,\mathbb{Z})}} d\gamma\, Z_{\text{mem}}(\gamma_{AB}; g_{MN}, C_{MNP})\,. \tag{2.2}$$

Here γ_{AB} is the constant metric on the world-volume 3-torus and g_{MN} and C_{MNR} the constant toroidal target-space metric and 3-form gauge field. In contrast to string amplitude (2.1), we expect to integrate over volume of the metric γ_{AB} because the membrane world-volume theory is not conformally invariant. We return to this important point later.

As a preliminary study, a semi-classical Polyakov quantization of the BPS membrane, based on a constant instanton summation measure, was performed in [8]. We found the correct instantonic saddle points: euclidean membranes wrapping all subtorii T^3 of the target space T^d, but with an incorrect summation measure, incompatible with U-duality. Similarly, the hamiltonian interpretation of the same result gave a consistent spectrum of BPS states running inside the loop, but with the wrong multiplicities. This result is hardly surprising, since the membrane world-volume theory exhibits U-duality symmetry at the classical level [17, 18], but quantum membrane self-interactions are expected to depend non-trivially on the instanton number. This was confirmed in a $\mathrm{U}(N)$ matrix model computation in [19], who found the correct instanton summation measure, in agreement with expectations from U-duality.

This failure of semi-classical quantization suggests a different approach: U-duality invariance should be manifestly built in from the beginning. To that end, observe that the string theory partition function is simply a standard gaussian theta series, invariant under the symplectic group $\mathrm{Sp}(2d,\mathbb{Z})$, which contains $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(d,d,\mathbb{Z})$ as commuting subgroups. The string theory moduli $g_{\mu\nu}, B_{\mu\nu}$ live on a slice of the symplectic period matrix moduli space according to the decomposition

$$\Big(\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)\Big)\times\Big(\mathrm{SO}(d,d,\mathbb{R})/[\mathrm{SO}(d)\times\mathrm{SO}(d)]\Big)\subset\mathrm{Sp}(2d)/\,\mathrm{U}(d)\,, \tag{2.3}$$

which is clearly preserved under the discrete $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(d,d,\mathbb{Z})$ T-duality subgroup.[2] More generally, the string winding mode partition function gives a correspondence between automorphic forms of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and the T-duality group $\mathrm{SO}(d,d,\mathbb{Z})$, by integrating with respect to the fundamental domain of either of the two factors. By analogy, we propose that the partition function of the membrane winding modes should be a theta series for a larger group including $\mathrm{SL}(3,\mathbb{Z})\times E_d(\mathbb{Z})$ as commuting subgroups. Specifically, we consider the following candidates

$$\begin{aligned} d=3 &\qquad \mathrm{GL}(3)\times[\mathrm{SL}(2)\times\mathrm{SL}(3)]\subset E_6\\ d=4 &\qquad \mathrm{GL}(3)\times[\mathrm{SL}(5)]\subset E_7\\ d=5 &\qquad \mathrm{GL}(3)\times[\mathrm{SO}(5,5)]\subset E_8\,. \end{aligned} \tag{2.4}$$

[2]For $d=2$, the group $\mathrm{Sp}(4)$ contain extra discrete generators which preserve this slice, which in particular can exchange the world-sheet and target space complex structures [20].

Here the GL(3) factor is to be interpreted as the world-volume modular group of a toroidal membrane (including the $\mathbb{R}^+$ volume factor) while the bracketed factor is the U-duality group E_d of d-toroidally compactified M-theory. The choice of the overarching group $E_{6,7,8}$ is the most economic guess, and will be justified *a posteriori* below. A prime reason to consider these groups is the fact that their minimal representation naturally involves cubic characters, which are expected when dealing with membrane Chern-Simons couplings.

According to our proposal, BPS supermembrane quantization therefore amounts to the construction of theta series for the exceptional groups $E_{6,7,8}$. The existence of exceptional theta series is known to mathematicians, but explicit expressions were unavailable until recently. In an earlier Article we have explicitly computed the generic summand in the theta series for any simply laced group G using techniques of representation theory [9]. The explicit summation measure and non-generic degenerate contributions have in turn been computed in [21]. An important feature of exceptional theta series is that invariance under the arithmetic subgroup $G(\mathbb{Z})$ does not reduce to the usual Poisson resummation of gaussian sums, but instead involves Fourier invariant cubic characters. This meshes nicely with the inherently cubic membrane couplings to the background 3-form C_{MNR}. Another important result from [9], is the requirement of extra flux-like quantum numbers over and above membrane winding numbers to realize the larger exceptional symmetries.

In this work, we concentrate on the $d = 3$ case, which has all the features of interest including membrane instantons, and carry out the integration in (2.2) using for $Z_{\rm mem}$ the theta series of E_6.

2.2 Exact R^4 amplitude in 8 dimensions

The four-graviton, eight-derivative amplitude in M-theory has received much attention in recent years, since it is determined completely non-perturbatively on the basis of supersymmetry and U-duality. The original conjecture by Green and Gutperle for type IIB string theory in ten dimensions [10] was later generalized to compactification on higher-dimensional tori T^d in [12]–[14]. For the case $d = 3$ of main interest in this Article, the result proposed in [12] reads

$$f_{R^4} = \widehat{E}^{\rm SL(3)}_{\mathbf{3},3/2}(G) + \widehat{E}^{\rm SL(2)}_{\mathbf{2},1}(\tau) \tag{2.5}$$

where the two terms are Eisenstein series for the two factors of the U-duality group $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$, namely

$$E^{\rm SL(3)}_{\mathbf{3},s} = \sum_{m^M \in \mathbb{Z}^3 \backslash \{0\}} \left[\frac{V^{2/3}}{m^M G_{MN} m^N} \right]^s , \qquad E^{\rm SL(2)}_{\mathbf{2},s} = \sum_{(m,n) \in \mathbb{Z}^2 \backslash \{0\}} \left[\frac{\tau_2}{|m + n\tau|^2} \right]^s , \tag{2.6}$$

where G_{MN} is the constant metric on the target space torus T^3 with volume $V = \det^{1/2} G_{MN}$, and τ is the volume modulus $\tau = C + i\frac{V}{l_M^3}$ complexified by the constant 3-form gauge field $C = C_{123}$ on T^3. The moduli take values in the product of symmetric spaces

$$(G_{MN}, \tau) \in [\mathrm{SO}(3) \backslash \mathrm{SL}(3,\mathbb{R}) / \mathrm{SL}(3,\mathbb{Z})] \times [\mathrm{SO}(2) \backslash \mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{Z})] , \tag{2.7}$$

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and transform under the left action of the U-duality group. The hat appearing on E denotes the fact that the automorphic forms appearing in (2.5) are really the finite terms following the pole at $s = 3/2$ (respectively $s = 1$),

$$E^{\mathrm{SL}(3)}_{\mathbf{3},s} = \frac{2\pi}{s-3/2} + \widehat{E}^{\mathrm{SL}(3)}_{\mathbf{3},3/2} + \mathcal{O}\left(s - \frac{3}{2}\right), \qquad E^{\mathrm{SL}(2)}_{\mathbf{2},s} = \frac{\pi}{s-1} + \widehat{E}^{\mathrm{SL}(2)}_{\mathbf{2},1} + \mathcal{O}(s-1)\,. \tag{2.8}$$

The conjecture (2.5) has been checked in many different ways:

1. The original motivation came from a perturbative analysis in type-IIA string theory compactified on T^2. From this viewpoint, the 3×3 matrix G_{MN} encodes the string coupling constant g_s and the complex structure U of the 2-torus, whereas the complex modulus τ corresponds to the complexified Kähler class T of the 2-torus. Upon expansion around the cusp at $g_s = 0$, the result (2.5) reproduces the tree-level and one-loop contributions in type-IIA compactified on T^2, together with an infinite set of non-perturbative effects attributed to euclidean D0-branes winding around the torus T^2:
$$f_{R^4} = \left[\frac{2\zeta(3)V}{g_s^2} - 2\pi \log U_2^4 |\eta(U)|^2 + f^{D0}_{R^4}\right] + \left[-2\pi \log T_2^4 |\eta(T)|^2\right]. \tag{2.9}$$
Here the two terms in brackets correspond to the contribution of the first and second term in (2.5), respectively. The summation measure for D0-instantons can be easily extracted from this result, and was successfully rederived from a matrix model computation in [22]. The second term arises purely at one-loop, and includes the effect of perturbative worldsheet instantons. Actually, the two terms above contribute to different kinematical structures $(t_8 t_8 \pm \epsilon_8\epsilon_8)R^4$, which we will not distinguish (they are identical in dimension $D \leq 7$). The vanishing of the R^4 amplitude at two loops implied by (2.5) has been recently confirmed [24] (see also [25] for a review of recent advances on two-loop technology).

2. The first term in (2.5) was derived up to an infinite additive constant from a one-loop computation in eleven-dimensional supergravity compactified on a two torus for which the integer summation variables $m^M \equiv m^{[1]}$ are Kaluza-Klein momenta of the graviton, or rather their canonical conjugates [26]. (This computation missed the second term, which can be attributed to membrane instantons wrapping T^3.) Indeed, expanding (2.5) around the cusp at $V \to \infty$, one obtains
$$\begin{aligned} f_{R^4} = {} & \frac{2\pi^2 V}{3 l_M^3} + \sum_{m^M \in \mathbb{Z}^3 \backslash \{0\}} \frac{V}{\left[m^{[1]} m_{[1]}\right]^{3/2}} - 2\pi \log\left(\frac{V}{l_M^3}\right) \\ & + \pi V \sum_{m^{[3]} \in \mathbb{Z} \backslash \{0\}} \frac{\mu(m^{[3]})}{\sqrt{|m^{[3]} m_{[3]}|}} \exp\left(-\frac{2\pi}{l_M^3}\sqrt{|m^{[3]} m_{[3]}|} + 2\pi i m^{[3]} C_{[3]}\right). \end{aligned} \tag{2.10}$$
The single summation integer $m^{[3]}$ can be viewed as a target space three-form[3] m^{MNP} which counts target 3-torus wrappings of an M2-brane with tension $1/l_M^3$. This is the

[3] So $m^{[3]} = m^{123}$ and $m^{[3]} m_{[3]} = V^2 (m^{123})^2$ where the volume factors come from three contractions with the target space metric G^{MN}.

main reason to expect that a one-loop supermembrane computation should reproduce membrane instantons and hence the full R^4 couplings if U-duality can be maintained. In this Article, we search for a derivation of this result in terms of fundamental super membrane excitations so that wrappings are expressed in terms of winding numbers

$$m^{MNP} = \frac{1}{3!}\epsilon^{ABC}\, Z_A^M Z_B^N Z_C^P\,. \tag{2.11}$$

The integer-valued matrix Z_A^M counts windings of the Ath world-volume cycle about the Mth target space one. The summation measure $\mu(m^{[3]})$ for membrane instantons on T^d is easily extracted from the SL(2) weight 1 Eisenstein series, and reads

$$\mu(m^{[3]}) = \sum_{n|m^{[3]}} n\,. \tag{2.12}$$

A semi-classical supermembrane computation [8] already yields the correct U-duality invariant exponent in (2.10) but does not correctly predict the counting of states given by $\mu(m^{[3]})$. It is an important challenge to rederive the result (2.10) including the correct measure factor from the membrane theory.

3. It was shown at the linearized level for M-theory on $d = 3$ [27] or at the non-linear level for type IIB [28] that supersymmetry requires the R^4 amplitude to be an eigenmode of the laplacian on the moduli space with a specific eigenvalue:

$$\Delta^{(2)}_{\mathrm{SO}(3)\backslash\,\mathrm{SL}(3)} f_{R^4} = \Delta^{(2)}_{\mathrm{SO}(2)\backslash\,\mathrm{SL}(2)} f_{R^4} = 0 \tag{2.13}$$

in the $d = 3$ case. It is also possible to check that the SL(3) part is annihilated by the cubic Casimir $\Delta^{(3)}_{\mathrm{SO}(3)\backslash\,\mathrm{SL}(3)}$, defined below in (3.23). In fact, the analysis in [27, 28] was not sensitive to possible holomorphic anomalies, and indeed, due to the subtraction of the pole in (2.8), the right-hand side of (2.13) picks up a non-vanishing constant value, 2π. This can be viewed as the result from logarithmic infrared divergences, as expected for R^4 couplings in 8 dimensions, in close analogy with gauge couplings in 4 dimensions [29].[4] Under mild assumptions on the behavior at small coupling, equation (2.13)together with U-duality invariance in fact imply the correctness of (2.5).

The result (2.5) is therefore on a very firm footing, and is one of the few exact non-perturbative M-theoretic results available. Its derivation from first principles is therefore a central problem of this field, to which we now turn.

3. E_6 minimal representation and theta series

As discussed in the Introduction, our contention is that the exact R^4 amplitude (2.5) follows from a one-loop computation in the BPS membrane theory, whose partition function is provided by the E_6 theta series. In this section we recall the motivation for this claim, review the construction of the E_6 theta series, and identify the physical parameters inside the moduli space of the E_6 theta series.

[4]Recall that in the terminology of [29], the holomorphic anomaly originates from the degenerate orbit, i.e., the contribution of Kaluza-Klein gravitons.

3.1 From modular and U-duality invariance to E_6

We posit that the correlator of four graviton vertex operators on a supermembrane of topology T^3, at leading order in momenta, reduces to the partition function for the membrane winding modes, except for a summation measure which incorporates the effects of non-abelian interactions for multiply wrapped membranes. The partition function of these winding modes should furthermore be invariant under $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ U-duality, together with $\mathrm{SL}(3,\mathbb{Z})$ modular transformations. Invariance under the two $\mathrm{SL}(3,\mathbb{Z})$ factors is automatic as soon as the membrane theory is invariant under world-volume and target space diffeomorphisms. Invariance under the $\mathrm{SL}(2,\mathbb{Z})$ factor is a highly non-trivial statement from the point of view of the membrane, although under double reduction it amounts to the T-duality symmetry of the type-II string. A natural way to realize this symmetry is to require invariance under the larger group E_6, through the two-stage decomposition

$$\mathrm{SL}(3) \times \left(\mathrm{SL}(2) \times \mathbb{R}^+\right) \times \mathrm{SL}(3) \subset \mathrm{SL}(3)^3 \subset E_6 \,. \tag{3.1}$$

Other possibilities exist, but only this one seems to lead to a natural membrane interpretation. One should therefore construct an automorphic theta series for E_6. This has been undertaken in [9, 21], which we review here for completeness:

3.2 Theta series, representations and spherical vectors

Given a group G, a theta series is constructed as follows (see e.g. [16] for a more complete discussion): Let ρ be a representation of G acting on functions of some space X. Let f be a function invariant under the action of the maximal compact subgroup $K \subset G$,

$$\rho(k) f = f \,, \qquad \forall k \in K \,. \tag{3.2}$$

For so called spherical representations, this function is unique and known as the spherical vector. Let δ be a distribution in the dual space of X, invariant under an arithmetic subgroup $G(\mathbb{Z}) \subset G$. Then the series

$$\theta(g) = \langle \delta, \rho(g) f \rangle \,, \tag{3.3}$$

is a function on G/K, invariant under right action by $G(\mathbb{Z})$, hence an automorphic form[5] for $G(\mathbb{Z})$. The term "theta series" is usually reserved for the case when ρ is the minimal representation, i.e. the one of smallest functional dimension. In particular, the minimal representation carries no free parameter (excepting A_n, which admits a one-parameter family of minimal representations, associated to the homogeneity degree s in Eisenstein series such as (2.6)). This is a desirable feature for applications to supermembrane physics. The minimal representation was constructed explicitly for all simply laced groups in [30], based on the quantization of maximally nilpotent coadjoint orbits. The spherical vector was

[5] To be precise, automorphic forms are usually required to be eigenfunctions of the laplacian and higher order invariant differential operators associated with the Casimirs of G. This may be achieved by requiring irreducibility of the representation ρ.

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computed in our earlier work [9], and the invariant distribution δ was obtained recently by adelic methods [21]. For example, when $G = \mathrm{Sp}(2,\mathbb{Z})$, one recovers the standard gaussian theta series, as follows: The minimal representation is the metaplectic representation acting on $X = L_2(\mathbb{R})$ with generators ∂_x^2, x^2 and $x\partial_x + \partial_x x$; the (quasi)spherical vector f is the gaussian $f = e^{-x^2}$, and δ is the comb distribution $\delta_{\mathbb{Z}}$, invariant under integer shifts and Fourier transform. The latter can be viewed adelically as the product over all primes p of the spherical vector f_p over the p-adic field $\mathbb{Q}_p$, which is simply the unit function on the p-adic integers, invariant under integer shifts and Fourier transform.

3.3 Minimal representation of E_6

We now turn to details of the E_6 theta series. Representations of non-compact groups are generally obtained by quantizing the coadjoint orbit of an element e in G^*. The representation of smallest dimension arises upon choosing a non-diagonalizable element e of maximal nilpotency, which can always be conjugated into the lowest root $E_{-\omega}$. The generators $E_{-\omega}, E_\omega$ and $H_\omega = [E_\omega, E_{-\omega}]$ form an SL(2) subalgebra, with maximal commuting algebra SL(6) in E_6,

$$\begin{aligned} E_6 &\supset \mathrm{SL}(2) \times \mathrm{SL}(6) \\ 78 &= (3,1) \oplus (2,20) \oplus (1,35)\,. \end{aligned} \tag{3.4}$$

The Cartan generator H_ω of this SL(2) subalgebra grades G into 5 subspaces which form representations of the commutant SL(6):

H_ω charge	2	1	0	-1	-2
SL(6) irreps	1	20	$1+35$	20	1
generators	E_ω	$\{E_\beta, E_\gamma\}$	$\{H_\omega, H_\beta, H_\gamma, H_\alpha, E_{\pm\alpha}\}$	$\{E_{-\beta}, E_{-\gamma}\}$	$E_{-\omega}$

(3.5)

The stabilizer S of $E_{-\omega}$ is given by the grade -2 and grade -1 subspace, together with the non-singlet part of the grade 0 subspace. Therefore, the coadjoint orbit $\mathcal{O} = S\backslash G$ of $E_{-\omega}$ can be parameterized by the orthogonal complement to the stabilizer S, namely the grade 1 and grade 2 spaces, together with the singlet in the grade 0 space. It carries the canonical Kirillov-Kostant symplectic structure and admits a right action of the group G. The action of G can thus be represented in terms of canonical generators acting by Poisson brackets. For E_6, we then have a representation on functions of a 22 dimensional, classical, phase space $\mathcal{O}$.

The minimal representation of E_6 acts on functions of half as many variables and is obtained by quantization of the classical system. The first step is to choose a polarization or lagrangian subspace of $\mathcal{O}$, i.e.: split the 22 coordinates into 11 positions and momenta. This is easily done by noting that the grade 1 subspace commutes to E_ω and hence forms a Heisenberg subalgebra. A particular choice of polarization is to break the symmetry SL(6) to a subgroup $H_0 = \mathrm{SL}(3) \times \mathrm{SL}(3)$ realized linearly on positions, which decomposes the grade 1 subspace as $20 = 1 + (3,3) + (3,3) + 1$. We choose as positions one of the two copies of $1 + (3,3)$ and call $(E_{\gamma_0}, E_{\gamma_A^M})$ position operators with $(E_{\beta_0}, E_{\beta_M^A})$ being their conjugate momenta. An extra position variable y corresponding to E_ω plays the rôle of $\hbar$. These generators are represented in the usual way

$$\begin{aligned} E_\omega &= iy\,, \\ E_{\beta_0} &= y\,\partial_{x_0}\,, \qquad E_{\gamma_0} = ix_0\,, \\ E_{\beta^A_M} &= y\,\partial_{Z^M_A}\,, \qquad E_{\gamma^M_A} = iZ^M_A\,. \end{aligned} \tag{3.6}$$

The remaining generators may be obtained using Weyl reflections as explained in [30, 9], and are

$$\begin{aligned}
E_\omega &= iy \\
E_{\beta_0} &= y\,\partial_{x_0} \\
E_{\beta^A_M} &= y\,\partial_{Z^M_A} \\
E_{\gamma_0} &= ix_0 \\
E_{\gamma^M_A} &= iZ^M_A \\
L^B_A &= -Z^M_A\partial_{Z^M_B} = -E_{-\alpha^A_B} \qquad \mathrm{SL}(3)_L \\
R^M_N &= -Z^M_A\partial_{Z^N_A} = -E_{-\alpha^N_M} \qquad \mathrm{SL}(3)_R \\
E_{\alpha^A_M} &= -x_0\partial_{Z^M_A} + \frac{i}{2y}\,\varepsilon^{ABC}\varepsilon_{MNR}Z^N_B Z^R_C \\
E_{-\alpha^M_A} &= Z^M_A\partial_{x_0} - \frac{iy}{2}\varepsilon_{ABC}\varepsilon^{MNR}\partial_{Z^N_B}\partial_{Z^R_C} \\
H_{\alpha^M_A} &= -x_0\partial_{x_0} + Z^M_A\partial_{Z^M_A} + (1-\delta_{AB})(1-\delta_{MN})Z^N_B\partial_{Z^N_B} + 2 \qquad \text{(no sum on } A,\, M) \\
H_{\beta_0} &= -y\partial_y + x_0\partial_{x_0} \\
H_\omega &= -2y\partial_y - x_0\partial_{x_0} - Z\cdot\partial_Z - 6 \\
E_{-\beta_0} &= -x_0\partial_y - \frac{i\det Z}{y^2} \\
E_{-\beta^A_M} &= Z^M_A\partial_y + \frac{i}{2}\,x_0\,\varepsilon_{ABC}\varepsilon^{MNR}\,\partial_{Z^N_B}\partial_{Z^R_C} + \frac{1}{y}\,(Z^M_A\,[Z\cdot\partial_Z+2] - Z^N_A Z^M_B\partial_{Z^N_B}) \\
E_{-\gamma_0} &= -y\det[\partial_Z] - i(y\partial_y + x_0\partial_{x_0} + Z\cdot\partial_Z + 6)\partial_{x_0} \\
E_{-\gamma^M_A} &= i(y\partial_y + x_0\partial_{x_0} + 4)\partial_{Z^M_A} + iZ^N_A\partial_{Z^N_A}\partial_{Z^M_B} + \frac{1}{2y}\,\varepsilon^{ABC}\varepsilon_{MNR}Z^N_B Z^R_C \\
E_{-\omega} &= -i(y\partial_y + x_0\partial_{x_0} + Z\cdot\partial_Z + 6)\partial_y + x_0\det[\partial_Z] - \\
&\quad - \frac{i}{y}\left(2Z\cdot\partial_Z + \frac{1}{2}(Z\cdot\partial_Z)^2 - \frac{1}{2}Z^M_A Z^N_B\partial_{Z^N_B}\partial_{Z^N_A} + 6\right) + \frac{\det(Z)}{y^2}\,\partial_{x_0}\,.
\end{aligned} \tag{3.7}$$

Pertinently, the generator for the negative root $-\beta_0$ involves the cubic H_0 invariant $I_3 \equiv \det Z$,

$$E_{-\beta_0} = -x_0\partial_y - \frac{i\det Z}{y^2}\,. \tag{3.8}$$

Functions invariant under the maximal compact subgroup $Usp(4)$ of E_6 must be annihilated by the compact generator

$$K_{\beta_0} = E_{\beta_0} + E_{-\beta_0} = y\,\partial_{x_0} - x_0\partial_y - \frac{i\det Z}{y^2}\,. \tag{3.9}$$

Recognizing the first two terms as an SO(2) rotation generator, this restricts the $Usp(4)$ invariant functional space to

$$f(y, x_0, Z^M_A) = \exp\left(i\frac{x_0 \det Z}{y(y^2+x_0^2)}\right)\hat{f}(y^2+x_0^2, Z^M_A)\,. \tag{3.10}$$

Automorphic E_6 invariance in this polarization of the minimal representation relies on (Fourier) invariance of the cubic character $\exp(i\det(Z)/x_0)$ (these are discussed further in [31]). Furthermore, in this scheme, the shifts of the target space three-form C can be identified with the action of the generator $E_{-\beta_0} = y\partial_{x_0}$, whose effect is to shift $x_0 \to x_0 + Cy$. The spherical functions thus couple to the C field by a *cubic* phase

$$\exp\left(i\frac{C \det Z}{y^2+[x_0+Cy]^2}\right). \tag{3.11}$$

reminiscent of the Chern-Simons coupling $\exp(iC\det Z)$ expected for a membrane. This cubic coupling is one of the main reasons to believe that E_6 can appear as an overarching group for the membrane on T^3. The appearance of additional variables (y, x_0) in the denominator is a new feature predicted by this E_6 symmetry.

To summarize, E_6 is represented on functions of 11 variables (y, x_0, Z^M_A) with $a = 1, 2, 3$ and $A = \dot{1}, \dot{2}, \dot{3}$. The generators of the $\mathrm{SL}(3) \times \mathrm{SL}(3)$ subgroup act linearly by matrix multiplication on $Z = (Z^M_A)$ from the left and right leaving y and x_0 invariant, while all remaining generators act non-linearly. The physical interpretation of these 11 variables will be discussed below.

3.4 E_6 spherical vector

The spherical vector f_{E_6} is invariant under the maximal compact subgroup $Usp(4)$ of E_6 generated by $K_\delta = E_\delta + E_{-\delta}$ for any positive root δ. So determining the spherical vector amounts to solving a complicated system partial differential equations $K_\delta f_{E_6} = 0$. Invariance under the maximal compact subgroup of the linearly acting $\mathrm{SL}(3)\times\mathrm{SL}(3)$ implies that f_{E_6} is a function of the quadratic, cubic and quartic invariants $\mathrm{tr}\, Z^{\mathrm{t}} Z$, $\det(Z)$, and $\mathrm{tr}(Z^{\mathrm{t}} Z)^2$. Invariance under the remaining compact generators fixes this function to be [9]

$$f_{E_6} = \frac{\exp(-S_1 - iS_2)}{(y^2+x_0^2)S_1}\,, \tag{3.12}$$

where

$$S_1 = \frac{\sqrt{\det[ZZ^{\mathrm{t}} + (y^2+x_0^2)\mathbb{I}\,]}}{y^2+x_0^2}\,, \qquad S_2 = -\frac{x_0\det(Z)}{y(y^2+x_0^2)}\,. \tag{3.13}$$

The exponential weight $S_1 + iS_2$ should be thought of as the classical action of a membrane with quantum numbers y, x_0, Z^M_A, at the origin of the moduli space $E_6/Usp(4) \supset [\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)]^3$. In the next section we explain how to couple the theory to worldvolume and target-space moduli. At this point, we note that the integers Z^M_A are naturally interpreted as windings arising from the zero-modes of transverse membrane coordinates:

$$X^M(\sigma^A) = Z^M_A \sigma^A + \cdots\,, \qquad Z^M_A = \partial_A X^M\,. \tag{3.14}$$

The action S_1 is a Born-Infeld type-generalization of the Polyakov membrane action.

The additional quantum numbers y, x_0 are necessary for manifest U-duality but cannot correspond to propagating membrane world-volume degrees of freedom which have already been accounted for by the windings Z_A^M. We propose, therefore, that they instead correspond to field strengths of a pair of two-form gauge fields B_{AB} and $B_{AB,MNR}$ (transforming as target space scalar and three-form densities), which have no propagating degrees of freedom in 3 dimensions, but whose field strengths take only quantized values.

Clearly if correct, our proposal, based on maintaining U-duality, provides a detailed microscope for examining fundamental membrane M-theory excitations. In particular, it would be interesting to extend U-duality invariance of the classical zero-mode action to quantum fluctuations. A remarkable feature of the E_6 spherical vector (3.12) is the simple exponential (corresponding to the Bessel function of index 1/2) which receives no quantum corrections about the classical action. In contrast, higher E_n cases involve genuine, "quantum corrected", Bessel functions of S_1.

3.5 Identification of physical parameters

We now present the decomposition of the extended duality group E_6 into world-volume modular and target space U-duality groups. Examining the E_6 Dynkin diagram in figure 1, we identify three commuting SL(3) subgroups with positive roots

$$\Delta^+_{\mathrm{SL}(3)_L} = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}\,, \qquad \Delta^+_{\mathrm{SL}(3)_R} = \{\alpha_{\dot{1}\dot{2}}, \alpha_{\dot{2}\dot{3}}, \alpha_{\dot{1}\dot{3}}\}\,, \qquad \Delta^+_{\mathrm{SL}(3)_{NL}} = \{\beta_0, -\omega, -\gamma_0\}\,. \tag{3.15}$$

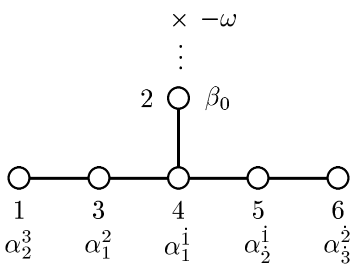
The first two factors $\mathrm{SL}(3)_L$ and $\mathrm{SL}(3)_R$ act linearly on Z_A^M by left and right multiplication, respectively. In line with our identification of Z_A^M as the winding numbers $\partial_A X^M$ of the membrane, one may thus associate $\mathrm{SL}(3)_L$ and $\mathrm{SL}(3)_R$ with the modular groups of the world-volume and target space 3-torii. The remaining SL(2) U-duality group, acting by fractional linear transformations on the complex modulus $\tau = C + i\frac{V}{l_M^3}$ (with $C \equiv C_{123}$) must reside in the remaining non-linearly acting $\mathrm{SL}(3)_{NL}$. A further decomposition

$$\mathrm{SL}(3)_{NL} \supset \mathrm{SL}(2)_\tau \times \mathbb{R}^+_\nu \tag{3.16}$$

allows the remaining $\mathbb{R}^+_\nu$ factor to be identified with the volume of the world-volume 3-torus. In the classical membrane theory, the membrane volume does not decouple because, in contrast to strings, the world-volume theory is not Weyl invariant.

One may wonder if this choice of parameterization is ambiguous. In particular, one may have considered identifying the world volume modular group with the non-linearly realized factor $\mathrm{SL}(3)_{NL}$. However, thanks to the triality symmetry of the extended E_6 Dynkin diagram in figure 1, one can show by using an appropriate intertwiner that any two of the $\mathrm{SL}(3)_{L,R,NL}$ groups can be made to act linearly. Identifying the world-volume modular group with the linearly realized $\mathrm{SL}(3)_L$ is convenient for our purposes, since we wish to integrate over the moduli space of world-volume metrics $\mathrm{SO}(3)\backslash\mathrm{SL}(3)$. Moreover, the combination of linearly acting world-volume and target-space modular groups is necessary for our membrane interpretation of the exceptional theta series.

To determine the decomposition of $\mathrm{SL}(3)_{NL}$ into world-volume and U-duality parts, we recall (as observed in section 3.3) that the target three-form couples through the exponential



$$
\begin{aligned}
\alpha^2_1 &= (0,0,1,0,0,0)\\
\alpha^3_2 &= (1,0,0,0,0,0)\\
\alpha^3_1 &= (1,0,1,0,0,0)\\
\alpha^{\dot 1}_{\dot 2} &= (0,0,0,0,1,0)\\
\alpha^{\dot 2}_{\dot 3} &= (0,0,0,0,0,1)\\
\alpha^{\dot 1}_{\dot 3} &= (0,0,0,0,1,1)
\end{aligned}
$$

$$
\begin{aligned}
\alpha^1_{\dot 1} &= (0,0,0,1,0,0)\\
\alpha^2_{\dot 1} &= (0,0,0,1,1,0)\\
\alpha^1_{\dot 2} &= (0,0,1,1,0,0)\\
\alpha^3_{\dot 1} &= (0,0,0,1,1,1)\\
\alpha^2_{\dot 2} &= (0,0,1,1,1,0)\\
\alpha^3_{\dot 2} &= (0,0,1,1,1,1)\\
\alpha^1_{\dot 3} &= (1,0,1,1,0,0)\\
\alpha^2_{\dot 3} &= (1,0,1,1,1,0)\\
\alpha^3_{\dot 3} &= (1,0,1,1,1,1)
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 &= (0,1,0,0,0,0) & \gamma_0 &= (1,1,2,3,2,1)\\
\beta^1_{\dot 1} &= (0,1,0,1,0,0) & \gamma^{\dot 1}_1 &= (1,1,2,2,2,1)\\
\beta^2_{\dot 1} &= (0,1,0,1,1,0) & \gamma^{\dot 2}_1 &= (1,1,2,2,1,1)\\
\beta^3_{\dot 1} &= (0,1,0,1,1,1) & \gamma^{\dot 3}_1 &= (1,1,2,2,1,0)\\
\beta^1_{\dot 2} &= (0,1,1,1,0,0) & \gamma^{\dot 1}_2 &= (1,1,1,2,2,1)\\
\beta^2_{\dot 2} &= (0,1,1,1,1,0) & \gamma^{\dot 2}_2 &= (1,1,1,2,1,1)\\
\beta^3_{\dot 2} &= (0,1,1,1,1,1) & \gamma^{\dot 3}_2 &= (1,1,1,2,1,0)\\
\beta^1_{\dot 3} &= (1,1,1,1,0,0) & \gamma^{\dot 1}_3 &= (0,1,1,2,2,1)\\
\beta^2_{\dot 3} &= (1,1,1,1,1,0) & \gamma^{\dot 2}_3 &= (0,1,1,2,1,1)\\
\beta^3_{\dot 3} &= (1,1,1,1,1,1) & \gamma^{\dot 3}_3 &= (0,1,1,2,1,0)
\end{aligned}
$$

$$\omega = (1,2,2,3,2,1)$$

Figure 1: Dynkin diagram and positive roots for E_6.

of the generator $e_{-\beta_0}$. Therefore we associate the non-linear SL(2) U-duality group with that generated by $\{E_{\beta_0}, H_{\beta_0}, E_{-\beta_0}\}$. The maximal commutant to the SL(2) $\subset$ SL$(3)_{NL}$ is the $\mathbb{R}^+$ group generated by

$$H_\nu = 2H_\omega - H_{\beta_0} = -3y\partial_y - 3x_0\partial_{x_0} - 2Z \cdot \partial_Z - 12\,. \tag{3.17}$$

From the expressions for the E_6 generators in eqs. 3.7 we read off the scaling weights of the various quantum numbers with respect to target and world volumes:

$$\begin{array}{|c|ccc|}\hline \text{quantum numbers} & y & x_0 & Z_A^M \\ \hline \text{target-space} & -1 & 1 & 0 \\ \text{world-volume} & -3 & -3 & -2 \\ \hline \end{array} \tag{3.18}$$

Therefore we choose the following couplings

$$y \to \nu V^{1/3} y\,, \qquad x_0 \to \nu V^{-1/3} x_0\,, \qquad Z \to \nu^{2/3} e^{-1} Z E\,, \tag{3.19}$$

where the 3×3 matrices

$$\gamma \equiv \nu^{2/3}\; ee^{\mathrm{t}}\,, \qquad G \equiv V^{2/3}\; EE^{\mathrm{t}}\,, \tag{3.20}$$

are the world-volume and target-space metrics, respectively. Note that although relative scalings of the two volumes amongst the variables (y, x_0, Z_A^M) are fixed, we will justify the overall ones by the results. We may now state our U-duality and world-volume modular invariant membrane winding formula

$$\begin{aligned}\theta_{E_6}(\gamma; G, C) = V\,\nu^2 \sum_{\substack{(y,x_0,Z)\\ \in \mathbb{Z}^{11}\backslash\{0\}}} \mu(y, x_0, Z) \frac{e^{-\frac{2\pi}{l_M^3}\frac{\sqrt{\det(ZGZ^{\mathrm{t}}+\gamma|x_0+\tau y|^2)}}{|x_0+\tau y|^2} - 2\pi i \det Z\left(\frac{x_0+Cy}{y|x_0+\tau y|^2}\right)}}{\sqrt{\det(ZGZ^{\mathrm{t}} + \gamma|x_0 + \tau y|^2)}} + \\ +\text{degen.}\end{aligned} \tag{3.21}$$

Salient features of this result include

- Although the variables Z_A^M of the minimal representation in eqs. 3.7 are real-valued, once integrated against the distribution δ in (3.3), they are restricted to integers. (This is also the origin of the overall factors 2π in the exponent.) Hence their natural interpretation as winding numbers.

- The real exponent is a Born-Infeld membrane action. It interpolates between Nambu-Goto (large membrane volume $V \gg \nu$) and Polyakov-like ($V \ll \nu$) actions.

- The membrane tension appears correctly as $1/l_M^3$ while the overall factor of the target space volume V matches correctly that of the bulk term in (2.10).

- The subleading degenerate terms and summation measure $\mu(y, x_0, Z)$ are known [21] and described in appendix A. The latter is derived from a p-adic analog of the spherical vector. It is a complicated number theoretic function representing the quantum degeneracies of winding states and was inaccessible to previous semi-classical approaches [8].

- The SL(2) modulus $\tau = C + i\frac{V}{l_M^3}$ coupling to the fluxes (y, x_0) may be written covariantly as $|H_{MNR} + C_{MNR}H|^2 + |H|^2$ where $x_0 \leftrightarrow H_{MNR} = dB_{MNR}$ and $y \leftrightarrow H \equiv dB$. The interpretation of x_0 as a target space 3-form is justified both by its coupling to the Chern-Simons 3-form and generalizations to higher torii discussed in section 5.

The simplest check of our proposal is whether it is a zero mode of the laplacian on the U-duality moduli space, as required by supersymmetry. This is the topic of the next section.

3.6 E_6 Casimirs

The desired R^4 amplitude (2.5) is a zero mode of the laplacian and invariant cubic operator of the $\mathrm{SL}(3)\times\mathrm{SL}(2)$ U-duality group. This is in fact separately true for the geometric target space SL(3) subgroup (and also the laplacian of the non-linear SL(2) factor with which we deal later). We can easily verify this property of our E_6 based R^4 amplitude as a simple consistency check: By virtue of the formula (3.3), relations valid for the enveloping algebra of the representation ρ apply also to the corresponding (differential) operators acting on moduli g. We therefore examine the quadratic and cubic Casimirs of the subgroups $\mathrm{SL}(3)_{L,R,NL}$,

$$\mathcal{C}_2 = \frac{1}{3!}\sum_{\alpha\in\Delta^+}\Big[H_\alpha^2 - 6\,E_\alpha E_{-\alpha}\Big]_{\mathrm{Weyl}}\,, \tag{3.22}$$

$$\mathcal{C}_3 = \frac{1}{2}\prod_{\alpha\in\Delta^+}\widetilde{H}_\alpha + \frac{9}{2}\left[\sum_{\alpha\in\Delta_+}\widetilde{H}_\alpha E_\alpha E_{-\alpha} - 3\sum_{\substack{\alpha,\beta,\gamma\in\Delta^+\\ \alpha+\beta=\gamma}}\Big(E_\alpha E_\beta E_{-\gamma} + E_{-\alpha}E_{-\beta}E_\gamma\Big)\right]_{\mathrm{Weyl}}\,. \tag{3.23}$$

Here $\alpha_{1,2}$ are simple roots and $\Delta^+ = \{\alpha_{1,2},\alpha_1+\alpha_2\}$ is the positive root lattice (see (3.15)). These compact expressions are "classical", the square brackets denote Weyl ordering averaging each term over all distinct orderings. Also, $\widetilde{H}_{\alpha_1}\equiv H_{\alpha_1}+H_{\alpha_1+\alpha_2}$, $\widetilde{H}_{\alpha_2}\equiv -H_{\alpha_2}-H_{\alpha_1+\alpha_2}$ and $\widetilde{H}_{\alpha_1+\alpha_2}\equiv H_{\alpha_1}-H_{\alpha_2}$. Inserting the explicit expressions for the minimal representation E_6 generators (see eqs. 3.7) yields invariant differential operators $\Delta^{(2,3)}_{\mathrm{SL}(3)_{R,L,NL}}$ subject to particular relations

$$\Delta^{(2)}_{\mathrm{SL}(3)_L} = \Delta^{(2)}_{\mathrm{SL}(3)_R} = \Delta^{(2)}_{\mathrm{SL}(3)_{NL}}\,,\qquad \Delta^{(3)}_{\mathrm{SL}(3)_L} = \Delta^{(3)}_{\mathrm{SL}(3)_R} = \Delta^{(3)}_{\mathrm{SL}(3)_{NL}}\,. \tag{3.24}$$

As explained above in (2.13), supersymmetry requires that the invariant target space operators $\Delta^{(2)}_{\mathrm{SL}(3)_R} = 0 = \Delta^{(3)}_{\mathrm{SL}(3)_R}$. However, upon integrating over the $\mathrm{SL}(3)_L$ fundamental domain of the world-volume torus shape moduli, the operators $\Delta^{(2,3)}_{\mathrm{SL}(3)_L}$ no longer act and must therefore return zero. Hence (3.24), in turn, implies vanishing of the target space invariants. As in the 1-loop string computation [29], infrared divergences may lead to holomorphic anomalies, and a constant non-vanishing right-hand side for (2.13). The explicit world-volume moduli integration is the subject of the following section.

4. Integrating over membrane world-volume moduli

An important difference between supermembranes and superstrings is the absence of a classical Weyl symmetry. An integral over all membrane volumes is, in general, divergent and must be appropriately regulated. On the other hand, integrations over SL(3) shape moduli of a world-volume 3-torus are better defined. This part of our calculation is completely analogous to its stringy counterpart: The summation over windings is decomposed into $\mathrm{SL}(3,\mathbb{Z})$ orbits and the integration over shape moduli can be performed by unfolding their fundamental domain.

An additional puzzle, at this stage, is the rôle of the flux-like quantum numbers (y, x_0). Our solution is to relate this difficulty to regulating the volume integral: Replacing the volume integral by one over additional compact moduli corresponding to shifts of the fluxes at the same time integrates out these additional quantum numbers while projecting out the dependence on the volume modulus. This procedure is a general technique in the theory of automorphic forms known as a constant term computation.[6] We begin the computation by integrating over shape moduli.

4.1 Integration over SL(3) shape moduli

The next step of our investigation of the conjecture that

$$Z_{\text{mem}}(\gamma_{AB}; G_{MN}, C_{MNP}) = \theta_{E_6}(\gamma_{AB}; G_{MN}, C_{MNP}) \,, \tag{4.1}$$

is to evaluate the modular integral over the $\mathrm{SO}(3)\backslash\mathrm{SL}(3)$ shape moduli of the world-volume 3-torus. Let us begin with a general description of the $\mathrm{SL}(3)_L$ modular integral:

4.1.1 Modular integral and orbit decomposition

The theta series summand at an arbitrary point of the moduli space $\mathrm{SO}(3)\backslash\mathrm{SL}(3)$, with Iwasawa gauge coset representative

$$e^{-1} = \begin{pmatrix} 1/L & & \\ & \sqrt{L/T_2} & \\ & & \sqrt{L\,T_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & A_1 & A_2 \\ & 1 & T_1 \\ & & 1 \end{pmatrix} , \tag{4.2}$$

is given in (3.21) depending on moduli e through (3.20). This summand was obtained by acting on the spherical vector with the linear $\mathrm{SL}(3)_L$ representation $\rho_L(e)$ acting by left multiplication on the windings Z^M_A. One is therefore left to compute

$$\theta_{\mathrm{SL}_3 \times \mathrm{SL}_3} = \int_{\mathcal{F}} de\, \theta_{E_6}(y, x_0, e^{-1}Z) \,. \tag{4.3}$$

The integration is over the fundamental domain $\mathcal{F} = \mathrm{SO}(3)\backslash\mathrm{SL}(3)_L/\mathrm{SL}(3,\mathbb{Z})$ of the moduli space of unit-volume constant metrics on T^3, with invariant measure

$$de = \frac{d^2T}{T_2^2}\, d^2A\, \frac{dL}{L^4} \,. \tag{4.4}$$

We lose no generality evaluating all other (target space) moduli at the origin. We stress again, that by construction $\theta_{\mathrm{SL}_3 \times \mathrm{SL}_3}$ is an automorphic form of the U-duality group $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$.

[6] An excellent review and useful results for SL(3) can be found in [32].

Modular integrals of this type can be computed by the general method of orbits: one restricts the summation on integers in θ_{E_6} to one representative in each orbit of the linear $\mathrm{SL}(3,\mathbb{Z})_L$ action, and at the same time enlarges the integration domain to the image of the fundamental domain under the $\mathrm{SL}(3,\mathbb{Z})$ orbit generators. Since Z transforms by $\mathrm{SL}(3,\mathbb{Z})$ left-multiplication, orbits are labeled by the rank of the 3×3 matrix Z. The non-degenerate orbits have $\mathrm{rank}(Z)=3$ and the fundamental domain can be enlarged to the full $\mathrm{SO}(3)\backslash\mathrm{SL}(3)$ moduli space. At the other end of the rank spectrum, the $\mathrm{rank}(Z)=0$ orbit contains the single element $Z=0$. The shape integral then yields only a factor of the volume of the fundamental domain $\mathcal{F}$. After integration over the volume factor $\mathbb{R}^+_\nu$, these two orbits will correspond to the Eisenstein series $E^{\mathrm{SL}(2)}_{\mathbf{2},1}$ in (2.5), hence to membrane instantons. The $\mathrm{rank}(Z)=1,2$ orbits require a detailed understanding of the fundamental domain $\mathcal{F}$. They correspond to toroidal, supergravity Kaluza-Klein excitations with amplitude $E^{\mathrm{SL}(3)}_{\mathbf{3},3/2}$. We deal with these terms first for which we are able to present a rigorous computation:

4.1.2 Rank 1 and 2 winding modes

To performing the integral over shape moduli it is convenient to rewrite the E_6 spherical vector in the integral representation

$$f_{E_6}=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{dt}{t^{1/2}}\exp\left(-\frac{1}{4t(y^2+x_0^2)^2}-t\det[ZZ^{\mathrm{t}}+(y^2+x_0^2)\mathbb{I}]-i\frac{x_0\det(Z)}{y(y^2+x_0^2)}\right). \quad (4.5)$$

Since $\mathrm{SL}(3)_L$ acts on the matrix Z by left multiplication, $Z\to e^{-1}Z$, it leaves the phase invariant. We may further set $x_0=0$ as it can be reinstated by an $\mathrm{SO}(2)$ rotation in the (y,x_0) plane.

We must now compute $\int_{\mathcal{F}_{\mathrm{rk}=1,2}}de\ f_{E_6}$ with respect to unfolded fundamental domains $\mathcal{F}_{\mathrm{rk}=1,2}$. The $\mathrm{SL}(3)$ fundamental domain $\mathcal{F}$ is known [33, 34], a complete description is given in appendix B.2, see in particular equations (B.19) and (B.21). Its construction is in terms of a height function given by the maximal abelian torus coordinate L^3 in (4.2) along with the actions of an (overcomplete) set of $\mathrm{SL}(3,\mathbb{Z})$ generators $\mathrm{S}_{1,\ldots,5}$, $\mathrm{T}_{1,2}$ and $\mathrm{U}_{1,2}$ given in (B.18), (B.20).

Rank 1 (twice degenerate) orbit: in this case the winding matrix Z may be rotated by an $\mathrm{SL}(3,\mathbb{Z})$ transformation from the right into a single row

$$Z=\begin{pmatrix}p_1 & p_2 & p_3\\ & & \\ & & \end{pmatrix}. \quad (4.6)$$

The range of the linear mapping Z is generated by a single $\mathrm{SL}(3)_L$ left-invariant vector transforming as a projective vector under the right $\mathrm{SL}(3)_R$ action. The matrix (4.6) is left invariant with respect to $\mathrm{SL}(3,\mathbb{Z})$ elements of the form

$$\begin{pmatrix}1 & * & *\\ & * & *\\ & * & *\end{pmatrix}, \quad (4.7)$$

spanned by generators S_2, $T_{1,2,3}$ and U_2. The conditions from the remaining generators may be unfolded yielding the integration range

$$\mathcal{F}_{\mathrm{rk}=1} = \Big\{ 0 \le T_1 \le \frac{1}{2}\,,\ T_1^2 + T_2^2 \ge 1\,,\ -\frac{1}{2} \le A_{1,2} \le \frac{1}{2}\,,\ 0 \le L < \infty \Big\}\,. \tag{4.8}$$

We find

$$\begin{aligned}\int_{\mathcal{F}_{\mathrm{rk}=1}} de\ f_{E_6} &= \frac{1}{\sqrt{\pi}} \int_{\mathcal{F}_{\mathrm{rk}=1}} de \int \frac{dt}{t^{1/2}}\ \exp\Big(-\frac{1}{4y^4 t} - t y^4 \Big[y^2 + \frac{p_1^2+p_2^2+p_3^2}{L^2} \Big] \Big) \\ &= \frac{1}{(p_1^2+p_2^2+p_3^2)^{3/2}} \frac{\pi}{3|y|} K_1(|y|)\,. \end{aligned} \tag{4.9}$$

The first factor is easily recognized as an Eisenstein series of $\mathrm{SL}(3)_R$ in the minimal representation, with index 3/2. On the other hand, reinstating the x_0 dependence dependence in the second term, $|y| \to \sqrt{y^2+x_0^2}$, yields the spherical vector (B.13) of the Eisenstein series of $\mathrm{SL}(3)_{NL}$ in the minimal representation, with index 0. Both of these can be identified with the general Eisenstein series (B.30) with $(\lambda_{32}, \lambda_{21})$ equal to $(-2,1)_R$ and $(1,1)_{NL}$, respectively. Hence, up to an overall normalization[7] we obtain, in the notation of appendix B.3,

$$\int_{\mathcal{F}} de\ \theta_{E_6}^{\mathrm{rk}=1} = E(g_R; -2, 1)\ E(g_{NL}; 1, 1)\,. \tag{4.10}$$

The $\mathrm{SL}(3)_R$ moduli are $g_R = EE^{\mathrm{t}}$, the unit, target space metric. We postpone dealing with the $\mathrm{SL}(3)_{NL}$ moduli g_{NL} to our discussion of the membrane world-volume integration in section 4.2.

Rank 2 (singly degenerate) orbit: the winding matrix Z may be rotated by an $\mathrm{SL}(3,\mathbb{Z})$ transformation into two rows,

$$Z = \begin{pmatrix} a_1 & p_1 & p_2 \\ & a_2 & p_3 \\ & & \end{pmatrix} \equiv \begin{pmatrix} \vec{v} \\ \vec{w} \\ \ \end{pmatrix}\,. \tag{4.11}$$

This configuration is invariant under the left $\mathrm{SL}(3,\mathbb{Z})$ action of elements of the form

$$\begin{pmatrix} 1 & & * \\ & 1 & * \\ & & * \end{pmatrix}\,. \tag{4.12}$$

These are spanned by generators $T_{2,3}$. All other generators may be unfolded yielding the integration range

$$\mathcal{F}_{\mathrm{rk}=2} = \Big\{ -\frac{1}{2} \le T_1\,,\ A_2 \le \frac{1}{2}\,,\ -\infty < A_1 < \infty\,,\ 0 \le T_2, L < \infty \Big\}\,. \tag{4.13}$$

[7] The rank 1 and 2 normalizations are computable, but unimportant since we have no such control over the rank 3 computation. Note also, there is in principle a six-fold ambiguity in the above identification, which is irrelevant due to the Selberg functional relations discussed in appendix B.1.

We find

$$\int_{\mathcal{F}_{\mathrm{rk}=2}} de\ f_{E_6} = \frac{1}{\sqrt{\pi}} \int_{\mathcal{F}_{\mathrm{rk}=2}} de \int \frac{dt}{t^{1/2}}\, e^{-\frac{1}{4y^4 t} - ty^2\left[y^4 + y^2\left(\frac{\lambda\vec{v}^2}{T_2} + \frac{(\vec{w}+A_1\vec{v}^2)}{L^2}\right) + \frac{|\vec{v}\times\vec{w}|^2}{LT_2}\right]}$$
$$= \frac{1}{|\vec{v}\times\vec{w}|^3}\frac{1}{y^2}\int_0^\infty \frac{dt\, dT_2\, dL}{t\, T_2^2\ L^3} e^{-\frac{1}{4ty^4} - ty^2\left(y^2+\frac{1}{\lambda^2}\right)\left(y^2+\frac{L}{T_2}\right)}\,. \tag{4.14}$$

Here the integrals over compact circles T_1 and A_2 are trivial and we performed the gaussian integral over A_1 explicitly. We now change variables

$$r_1 = L^2\,, \qquad r_2 = \frac{L}{T_2}\,, \tag{4.15}$$

and rescale $r_i \to r_i y^2$, $t \to t/y^5$, obtaining

$$\int_{\mathcal{F}_{\mathrm{rk}=2}} de\ f_{E_6} = \frac{y^3}{2|\vec{v}\times\vec{w}|^3} \int_0^\infty \frac{dt}{t}\sqrt{r_1}dr_1 dr_2\, e^{-\frac{y}{4t} - ty(1+r_1)(1+r_2)}\,. \tag{4.16}$$

The integral over r_2 is of Gamma function type. Carrying out the Bessel-type integration with respect to t and changing variable to $r_1 = u^2 - 1$, we find

$$\int_{\mathcal{F}_{\mathrm{rk}=2}} de\ f_{E_6} = \frac{4y^2}{|\vec{v}\times\vec{w}|^3} \int_1^\infty K_1(uy)\sqrt{u^2-1}\, du$$
$$= \frac{2\pi e^{-y}}{|\vec{v}\times\vec{w}|^3}\,. \tag{4.17}$$

Comparing again to (B.30) and (B.13), we recognize the product of $\mathrm{SL}(3)_{NL}$ and $\mathrm{SL}(3)_R$ continuous representations with parameters $(\lambda_{32}, \lambda_{21}) = (1,1)$, $(1,-2)$, respectively. Hence the rank 2 result is a product of corresponding SL(3) Eisenstein series. However, as explained in appendix B.4, minimal parabolic Eisenstein series obey Selberg relations, which amount to invariance[8] under (Weyl group) permutations of the labels $\lambda_1, \lambda_2, \lambda_3$. These are generated by reflections about radial lines $\lambda_{21} = \lambda_{32}$, $2\lambda_{21} = -\lambda_{32}$ and $\lambda_{21} = -2\lambda_{32}$ in the $(\lambda_{32}, \lambda_{21})$-plane depicted in figure 2. In particular, this implies, up to normalization, equality of the rank 2 and rank 1 winding sums. Hence

$$\int_{\mathcal{F}} de\,\left(\theta_{E_6}^{\mathrm{rk}=1} + \theta_{E_6}^{\mathrm{rk}=2}\right) = E(g_{NL}; -2, 1)\, E(g_R; -2, 1)\,. \tag{4.18}$$

Observe from figure 2 that the point $(-2,1)$ corresponds to vanishing quadratic and cubic Casimirs as predicted in section 3.6.

4.2 Integration over membrane volume — degenerate orbits

Before dealing with rank 3 and 0 winding sums, we study the rank 2 and 1 results and learn how to handle the membrane world volume integral. The lack of conformal (Weyl) invariance of the classical supermembrane theory has been a key stumbling block, intimately related to its gapless, continuous spectrum. This is precisely the sector where we expect to find new physics.

[8] The classical example is the SL(2) relation $E_s^{\mathrm{SL}(2)} \propto E_{1-s}^{\mathrm{SL}(2)}$. Equality holds for appropriate normalization by a function of s, see (B.45).

We start with an Iwasawa decomposition of $\mathrm{SL}(3)_{NL}$ moduli refined to exhibit the $\mathrm{SL}(2)$ U-duality subgroup:

$$g_{NL} = \begin{pmatrix} \frac{1}{\nu^{2/3}} & & \\ & \nu^{1/3} & \\ & & \nu^{1/3} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{\tau_2}} & \\ & & \sqrt{\tau_2} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \tau_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & n_1 & n_2 \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{4.19}$$

The membrane world volume modulus ν and target space moduli (τ_1, τ_2) were already introduced in section 3.5. There are, however, two additional possible moduli (n_1, n_2). Automorphy implies that these are compact unit interval valued variables. In contrast, the world volume modulus is non-compact taking any real positive value. In a (semi)classical supermembrane setting, the moduli (n_1, n_2) have no particular meaning and should be set to zero; one would further have to integrate over all world volumes $\nu \in \mathbb{R}^+$, which is an ill-defined non-compact integral. In particular this integral can be performed explicitly for the rank 1 and 2 winding sums (4.18) and seen to be divergent (see appendix C for a sample computation).

Instead we propose exchanging the world-volume modulus ν for the compact moduli (n_1, n_2). An integral over these moduli is well defined, and known as a constant term computation in the mathematical literature. Physically, we could then view (y, x_0) as auxiliary quantum numbers, necessary for U-duality based on a hidden exceptional symmetry group. They are "integrated out" by performing the (n_1, n_2) integrals. Indeed, (y, x_0) and (n_1, n_2) are world volume canonical conjugates. Essentially we are adding auxiliary world volume fields to make the exceptional symmetry manifest, and in turn integrating them out. The constant term result is then invariant under a subgroup $\mathrm{SL}(2)$, the Levi part of the parabolic group P_2 with unipotent radical spanned by (n_1, n_2).

Placing faith in our proposal we must now compute $\int_0^1 dn_1 dn_2 E(g_{NL}; -2, 1)$. Physicists can easily perform this computation by using the sum representation of the Eisenstein series and Poisson resumming, i.e. a small radius expansion in one direction [14]. A general mathematical machinery involving p-adic integrations has been developed for these computations by Langlands [35] (again a useful account is given in [32] and for completeness these results are reproduced in appendix B.4). Using (B.50), the part of the result required here is

$$\int_0^1 dn_1 dn_2 E(g_{NL}; 1-2s, 1) = \frac{\pi}{\zeta(3)} \frac{1}{s - 3/2} + \text{analytic}\,. \tag{4.20}$$

In this normalization the leading behavior at $s = 3/2$ is a simple pole. Importantly it is ν-independent! The subleading analytic terms[9] depend both on the volume modulus ν and the log of the Dedekind eta function of the U-duality modulus τ. Nonetheless, we claim

[9]In fact, since we have not studied the overall normalization, one might argue that the leading contribution in the product of Eisenstein series in (4.18) is a double pole with constant coefficient. The subleading single pole coefficients then include the Dedekind eta in the U-duality modulus τ. This is in principle a feasible situation since this is the correct result in the bulk wrapping sector. However, since there is anyway an implicit additive renormalization of the R^4 conjecture (2.5), we prefer the above presentation.

that the correct prescription is to keep the coefficient of the simple pole,

$$\int d^2n \int_{\mathcal{F}} de\, \left(\theta_{E_6}^{\mathrm{rk}=1} + \theta_{E_6}^{\mathrm{rk}=2}\right) \propto E^{\mathrm{SL}(3)}_{\mathbf{3},3/2}(\widehat{G})\,. \tag{4.21}$$

It remains only to derive the SL(2) Eisenstein series part of the the R^4 amplitude (2.5), from the rank 0 and rank 3 orbits.

4.3 Membrane wrapping sum

The rank 0 and 3 contributions to the membrane winding summation are independent of the unit $\mathrm{SL}(3)_R$ target space metric moduli and correspond to the summation over membrane wrappings in (2.10). Indeed the determinant of the winding matrix Z counts toroidal wrappings and corresponds to $m^{[3]}$ there. The rank 0 contribution amounts to setting $Z = 0$ and since no unfolding is possible, simply returns the volume of the fundamental domain $\mathcal{F}$. A *tour de force* calculation would track these degenerate contributions by taking account also those of the original E_6 theta series in (3.21) (described in appendix A). Here we are rather less ambitious, however, since the final result is guaranteed to be U-duality invariant, these terms are anyway fixed by automorphy and we will not consider them further. That leaves only the (most difficult) rank 3 terms:

Although we have gathered a great deal of information about the non-degenerate rank 3 term, unlike the lower rank contributions, we are unable to compute it exactly. We are able to (i) calculate an approximate expression for the $\mathrm{SL}(3)_{NL}$ spherical vector appearing after performing membrane shape moduli integrals; (ii) identify the underlying (novel) $\mathrm{SL}(3)_{NL}$ representation on which this spherical vector is based; (iii) compute the intertwiner between this representation of $\mathrm{SL}(3)_{NL}$ and the continuous series representation of SL(3); (iv) compute the action of this intertwiner on the approximate spherical vector although this does not yield an unambiguous identification of the $\mathrm{SL}(3)_{NL}$ automorphic form at hand. Points (i)-(iv) are presented chronologically in what follows. As evidence that upon integration over compact moduli (n_1, n_2) the result is the required SL(2) Eisenstein series, we employ the intertwiner of point (iii) to study the constant computation for the standard induced representation and show that the leading contribution is the correct one.

4.3.1 Rank 3 winding modes

An element Z of the non-degenerate orbit can be rotated by an $\mathrm{SL}(3,\mathbb{Z})$ matrix into

$$Z = \begin{pmatrix} a_1 & p & q \\ & a_2 & r \\ & & a_3 \end{pmatrix} \tag{4.22}$$

No $\mathrm{SL}(3,\mathbb{Z})$ generators leave this orbit representative invariant, so the fundamental domain can be completely unfolded to

$$\mathcal{F}_{\mathrm{rk}=3} = \Big\{ -\infty < T_1, A_1, A_2 < \infty\,,\, 0 \le T_2, L < \infty \Big\}\,. \tag{4.23}$$

We again employ the integral representation of the spherical vector (4.5) and the change of world volume shape variables (4.15). The integral with respect to $A_{1,2,3}$ is gaussian, leading to

$$\int \frac{r_1 dr_1 dr_2 dt}{t^2} \frac{\exp\left(-\frac{1}{4ty^4} - t\frac{(a_1^2 r_1+y^2)(a_2^2 r_1+y^2)(a_3^2+y^2 r_1 r_2)}{r_1 r_2}\right)}{r_1^{1/2} a_2 a_3^2 y^3 \sqrt{(a_1^2 r_1 + y^2)(a_2^2 r_1 + y^2)(a_3^2 + y^2 r_1 r_2)}} . \tag{4.24}$$

The integral over r_1, r_2, t can be performed in the saddle point approximation, and gives

$$f_{\mathrm{SL}(3)} = \int_{\mathcal{F}_{\mathrm{rk}=3}} def_{E_6} = \frac{\exp\left[-\frac{(y^2+x_0^2+x_1^2)^{3/2}}{y^2+x_0^2} - i\frac{x_0 x_1^3}{y(y^2+x_0^2)}\right]}{x_1^5 (y^2+x_0^2+x_1^2)^{1/4}} \tag{4.25}$$

where $x_1^3 \equiv a_1 a_2 a_3 = \det Z$ is the wrapping number. This saddle point result for the spherical vector becomes exact in the limit where (y, x_0, x_1) are scaled to infinity at the same rate. The fact that the result depends on the determinant of the matrix Z is guaranteed by $\mathrm{SL}(3)_L$-invariance, and implies that the result is an $\mathrm{SL}(3)_R$ singlet, as it should if it is to reproduce the SL(2) part in (2.5) after integration over the volume factor. The representation under $\mathrm{SL}(3)_{NL}$ is however more tricky to identify.

4.3.2 Representation of the non-degenerate orbit under $\mathrm{SL}(3)_{NL}$

Beginning with the E_6 minimal representation in eqs. 3.7, we can obtain the representation of $\mathrm{SL}(3)_{NL}$ by studying the action of the generators $(E_{\pm\omega}, E_{\pm\beta_0}, E_{\pm\gamma_0}, H_{\beta_0}, H_{\gamma_0})$ restricted to functions $\varphi(y, x_0, z \equiv x_1^3 = \det(Z))$. We find

$$\begin{aligned}
E_{\beta_0} &= y\partial_{x_0}\,, \qquad E_{-\beta_0} = -x_0\partial_y - \frac{iz}{y^2}\,,\\
E_{\gamma_0} &= ix_0\,, \qquad E_{-\gamma_0} = -i(6 + x_0\partial_{x_0} + y\partial_y + 3z\partial_z)\partial_{x_0} - y(6 + z^2\partial_z^2 + 6z\partial_z)\partial_z\,,\\
E_\omega &= iy\,, \qquad E_{-\omega} = -i(6 + x_0\partial_{x_0} + y\partial_y + 3z\partial_z)\partial_y + x(6 + z^2\partial_z^2 + 6z\partial_z)\partial_z -\\
&\qquad\qquad - \frac{3i}{y}\,(2 + z^2\partial_z^2 + 4z\partial_z) + \frac{1}{y^2}\,z\partial_y\,,\\
H_{\beta_0} &= x_0\partial_{x_0} - y\partial_y\,, \qquad H_{\gamma_0} = -2x_0\partial_{x_0} - y\partial_y - 3z\partial_z - 6\,.
\end{aligned} \tag{4.26}$$

In section 3.6 we argued that upon integrating out $\mathrm{SL}(3)_L$ moduli, the remaining $\mathrm{SL}(3)_{R,NL}$ Casimir invariants should vanish. Indeed a straightforward computation yields

$$\Delta^{(2)}_{\mathrm{SL}(3)_{NL}} = 0 = \Delta^{(3)}_{\mathrm{SL}(3)_{NL}}\,. \tag{4.27}$$

The spherical vector for the representation (4.26) may in principle be computed by solving the partial differential equations associated to the maximal compact subgroup SO(3). We are not able to integrate these equations exactly, however the leading result in the limit where y, x_0, x_1 are scaled to infinity simultaneously agrees with (4.25).

4.3.3 Intertwiner from cubic to induced representations

We now identify the novel "cubic" $\mathrm{SL}(3)_{NL}$ representation found in (4.26). As we recall in appendix B.1, all continuous irreducible representations of SL(3) can be obtained by induction from the minimal parabolic subgroup P of lower triangular matrices, with a character (B.2). These are natural candidates so long as the parameters $(\lambda_{32}, \lambda_{21})$ lie at the intersection of vanishing quadratic and cubic Casimir loci depicted in figure 2:

$$(\lambda_{32}, \lambda_{21}) \in \{(-1,2),(1,1),(2,-1),(1,-2),(-1,-1),(-2,1)\}\,. \tag{4.28}$$

Note that these six solutions are related by action of the Weyl group, hence the corresponding Eisenstein series by Selberg's relations (B.49). We begin therefore with the representation (B.6) and search for an intertwiner bringing it to the form (4.26). Let us now perform a few changes of variables: We first Fourier transform over v, w, and write $\partial_w = ix_0, \partial_v = iy$. The generator $-E_\gamma$ becomes $y\partial_0 + \partial_x \equiv E_{\beta_0}$. Similarly we identify E_β with E_{γ_0}. In order to get rid of the ∂_x term, we redefine

$$x \to x_1 + \frac{x_0}{y}\,, \qquad \partial_x \to \partial_1\,, \qquad \partial_0 \to \frac{\partial_0 - \partial_1}{y}\,, \qquad \partial \to \frac{\partial + x_0\partial_1}{y^2}\,. \tag{4.29}$$

The generator H_{β_0} now becomes $-y\partial + x_0\partial_0 + 2x_1\partial_1$. We eliminate the last term by further redefining

$$x_1 = \frac{x_2}{y^2}\,, \qquad \partial_1 \to y^2\partial_2\,, \qquad \partial_0 \to \partial_0\,, \qquad \partial \to \partial + \frac{2x_2\partial_2}{y}\,. \tag{4.30}$$

The generator $E_{-\gamma_0}$ now reads $-x_0\partial + x_2^2\partial_2/y^2$. We put $x_2 = 1/x_3$ so that

$$E_{-\beta_0} = -x_0\partial - \frac{\partial_3}{y^2} + (1-\lambda_{32})\left(-\frac{x_0}{y} - \frac{1}{x_3 y^2}\right). \tag{4.31}$$

Only when $\lambda_{32} = 1$, the singular term $1/(x_3 y^2)$ disappears, so we may Fourier transform one last time over x_3 and write $\partial_3 = -iz$. This yields a one-parameter family of SL(3) representations [16] depending on λ_{32}. Setting also $\lambda_{21} = 1$, we obtain precisely the representation (4.26). We have thus identified the $\mathrm{SL}(3)_{NL}$ representation arising by integrating the E_6 theta series over the action of $\mathrm{SL}(3)_L$ at a generic (rank 3) point, with the Eisenstein series of $\mathrm{SL}(3)_{NL}$ with parameters $(\lambda_{32}, \lambda_{21}) = (1,1)$,

$$\int_{\mathcal{F}} de\; \theta_{E_6}^{\mathrm{rk}=3} \sim E(g_{NL}; 1, 1)\,. \tag{4.32}$$

Because the point $(1,1)$ lies at the intersection of two lines of single poles, it is not clear however whether the right-hand side should be understood as the residue, or whether some finite term should be kept — we will return to this point shortly.

4.3.4 Intertwining the spherical vector

We begin with the known spherical vector (B.9) of the continuous representation computed in appendix B.1. Because $\lambda_{32} = \lambda_{21} = 1$ in the intertwined $\mathrm{SL}(3)_{NL}$ representation, we must study the limit $s = t = 0$. We may however regard non-zero s and t as a regulator. In particular, representing the exact continuous representation spherical vector as

$$f_{\mathrm{SL}(3)} = \frac{\pi^{s+t}}{\Gamma(s)\Gamma(t)} \int_0^\infty \frac{dt_1 dt_2}{t_1^{1+s} t_2^{1+t}} \exp\left(-\frac{\pi(1+x^2+(v+xw)^2)}{t_1} - \frac{\pi(1+v^2+w^2)}{t_2}\right), \tag{4.33}$$

and intertwining to the representation (4.26) (i.e. performing the string of Fourier transforms and variable changes of the preceding section), in the saddle approximation we recover the action appearing in the exponent of (4.25). The final t_2 integration yields an overall infinite factor, however. This situation is reminiscent of the spherical vector for the maximal parabolic Eisenstein series which can be obtained by a single Fourier transform over the variable w. The rationale being that the continuous representation induced from the minimal parabolic with abelian character (B.2), is equivalent to that induced from the maximal parabolic with a non-trivial representation in the SL(2) Levi subgroup. The possibility of attaching cusp forms correspondint to discrete representations to the Levi factor, does lead to independent Eisenstein series, however [36]. Due to this type of subtlety, we cannot unambiguously identify the rank 3 winding sum with a minimal parabolic Eisenstein series at $\lambda_{32} = \lambda_{21} = 1$.

4.4 Integration over membrane volume — non-degenerate orbit

We now return to the issue of integrating over the membrane volume $\nu \in \mathbb{R}^+$. As we argued above, the correct prescription is to compute the constant term corresponding to the maximal parabolic group P_2 generated by (n_1, n_2). Physically, this amounts to computing the Fourier coefficients associated to (n_1, n_2) at zero momentum, or equivalently, averaging over the action of E_{γ_0} and E_ω. Since (y, x_0) are the conjugate variables (from (4.26)), one should evaluate the spherical vector at the origin $y = x_0 = 0$. Unfortunately, this is the limit where the saddle point approximation used to derive (4.25) breaks down. This is also the limit in which the degenerate contributions are obtained, on which we have no control.

A better strategy is therefore to intertwine back to the standard induced representation (B.6), for which constant terms are completely known. Having just established that the $\mathrm{SL}(3)_{NL}$ representation (4.26) is equivalent to the induced representation $(-1, -1)$, it remains to establish how the double pole at $(-1, -1)$ should be regularized. Unfortunately, we have not been able to settle this point, however it is easy to find an ad hoc prescription which gives the correct result: expanding the constant term computation (B.47) around $\lambda_{21} = -1 + \epsilon_1, \lambda_{32} = -1 + \epsilon_2$, we have

$$E_{P_2} = 2t_1^{3-\epsilon_1-\frac{1}{2}\epsilon_2}\zeta(2-\epsilon_2)E^{\mathrm{SL}(2)}_{\mathbf{2},1-\frac{1}{2}\epsilon_2} + 2t_1^{\frac{3}{2}+\frac{1}{2}\epsilon_1-\frac{1}{2}\epsilon_2}\zeta(3-\epsilon_1-\epsilon_2)\frac{\xi(1-\epsilon_1)}{\xi(\epsilon_1-1)}E^{\mathrm{SL}(2)}_{\mathbf{2},3/2-\frac{1}{2}\epsilon_2-\frac{1}{2}\epsilon_2} + \\ +2t_1^{\frac{1}{2}\epsilon_1+\epsilon_2}\zeta(2-\epsilon_1)\frac{\xi(1-\epsilon_2)\xi(2-\epsilon_1-\epsilon_2)}{\xi(\epsilon_2-1)\xi(\epsilon_1+\epsilon_2-2)}E^{\mathrm{SL}(2)}_{\mathbf{2},3/2-\frac{1}{2}\epsilon_2-\frac{1}{2}\epsilon_2}\,. \tag{4.34}$$

In this formula, the $\mathbb{R}^+_\nu$ modulus is related to the membrane volume by $t_1 = \nu^{1/6}$. Picking the residue of the $1/\epsilon_2$ pole, extracting from it the finite term following the $1/\epsilon_1$ pole, and using the relation $\zeta'(2) = -\zeta(3)/(4\pi^2)$, we find

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_2\partial_{\epsilon_1}\epsilon_1 E_{P_2} = \frac{2\pi^3}{3\zeta(3}\widehat{E}^{\mathrm{SL}(2)}_{\mathbf{2},1-\frac{1}{2}\epsilon_1} + \frac{2\pi^4}{3\zeta(3)}\log t_1 + \frac{\pi^2}{3}t_1^3 + \mathrm{cste}\,. \tag{4.35}$$

The volume independent term does indeed reproduce the SL(2) Eisenstein series from (2.5)! The meaning of the other terms is however far from clear. It would be interesting to obtain a detailed understanding of the singularity at $(\lambda_{32}, \lambda_{21}) = (1, 1)$ from the point of view of the membrane computation.

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5. Membranes on higher dimensional torii, and membrane/five-brane duality

Having extolled our present understanding of the consequences of the E_6 theta series conjecture for M-theory on T^3, we now briefly discuss the higher dimensional generalization of our construction, and present a provocative hint that our framework may incorporate membrane/fivebrane duality.

5.1 BPS membranes on T^4

Let us first discuss the generalization of our construction to the case of M-theory compactified on T^4-similar considerations can be applied to E_8 and T_5 compactifications. Here, we expect a symmetry under the U-duality group SL(5), which contains the obvious geometrical symmetry SL(4) of the target 4-torus, together with U-duality reflections which invert the volume of a sub 3-torus,

$$R_M \to \frac{l_M^3}{R_N R_P}\,, \qquad l_M^3 \to \frac{l_M^6}{R_M R_N R_P} \tag{5.1}$$

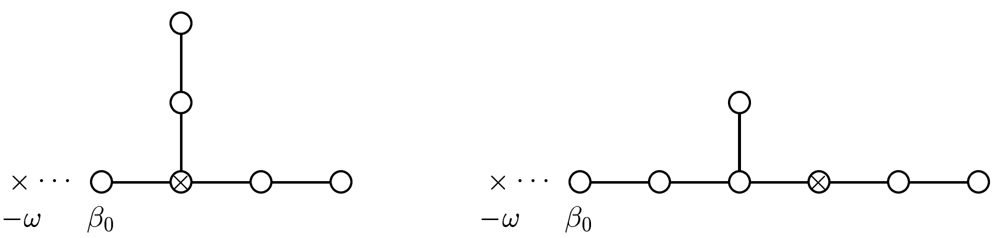
for any choice of 3 directions (M, N, P) out of 4. The R^4 couplings have been argued in [12] to be given by an SL(5) Eisenstein series of weight 3/2.

Just as for T^3, we expect this amplitude to be derivable from a one-loop amplitude in membrane theory, i.e., an integral over the partition function describing minimal maps $T^3 \to T^4$. In order to that the symmetry under $\mathrm{SL}(3) \times \mathrm{SL}(5)$ be non-linearly realized, we assume an overarching symmetry under $E_{7(7)}(\mathbb{Z})$ in the minimal representation, which now has dimension 17. The canonical presentation of the minimal representation has a linearly realized SL(6) subgroup, under which the 17 variables are arranged as a 6×6 antisymmetric matrix X, and two singlets (y, x_0, X). The spherical vector has been worked out in [9], and, for large quantum numbers, becomes an exponential of minus the action

$$S = \frac{\sqrt{\det(X + |z|\mathbb{I}_6)}}{|z|^2} - i\,\frac{x_0 \mathrm{Pf(X)}}{y|z|^2}\,, \tag{5.2}$$

where $z \equiv y + ix_0$.

This action however does not have a direct intepretation in terms of T_4 winding modes, which would make a 3×4 matrix of integers. It is however possible to make a judicious choice of polarization for the Heisenberg subalgebra where $\mathrm{SL}(3)\times\mathrm{SL}(4)$ is linearly realized, as follows. Let us examine the E_6 and E_7 extended Dynkin diagrams



(5.3)

The longest root $-\omega$ is denoted by a $\times$. The node labeled β_0 determines the Levi subalgebra acting linearly on the Heisenberg subalgebra (positions and momenta). It is obtained by deleting nodes $-\omega$ and β_0 which yields SL(6) and SO(6, 6) groups, respectively. The choice of a set of momenta and coordinates, i.e., a polarization, breaks these groups to a subgroup. The node marked with a cross determines a choice of polarization appropriate for membrane winding sums. The remaining nodes are subgroups linearly acting on positions, namely $\mathrm{SL}(3)\times\mathrm{SL}(3)$ and $\mathrm{SL}(3)\times\mathrm{SL}(4)$-precisely the linear actions from left and right on 3 and 4-torus winding matrices. Note that only for E_6 does the canonical polarization determined by the node attached to β_0 coincide with the membrane inspired one. For E_8, 5-torus winding sums, more general choices of β_0 are even necessary.

Deleting the node marked with a cross from the extended Dynkin diagrams, leaves the U-duality and world-volume groups. We have discussed the E_6 case in detail. For E_7, the two rightmost nodes correspond to the SL(3) membrane world-volume shape moduli. The nodes remaining on the left form $\mathrm{SL}(6)\supset\mathrm{SL}(5)\times\mathbb{R}_+$, a product of the T_4 U-duality group and the world-volume. (For T_4, the integration over world-volumes is also regulated by quantum fluxes.) Also, just as E_6 has a triality, duality of the E_7 Dynkin diagram implies that there is no ambiguity in the choice of linearly acting, SL(3), world-volume modular group.

By looking at the root lattice, we may now determine the transformation properties of the seventeen variables of the minimal representation with respect to the linearly acting $\mathrm{SL}(3)\times\mathrm{SL}(4)$ subgroup: the singlet y remains unaffected, while x_0 and the elements of the 6×6 antisymmetric matrix X reassemble themselves into a 4×3 matrix Z and a 4-vector X,

$$X=\begin{pmatrix} X_{12}\\ X_{31}\\ X_{23}\\ x_0\end{pmatrix},\qquad Z=\begin{pmatrix} X_{14} & X_{15} & X_{16}\\ X_{24} & X_{25} & X_{26}\\ X_{34} & X_{35} & X_{36}\\ y\partial_{X_{56}} & y\partial_{X_{64}} & y\partial_{X_{45}}\end{pmatrix}. \tag{5.4}$$

The derivatives in the last row indicate that a Fourier transform should be performed in these variables before the symmetry becomes linearly realized. The spherical vector in this new polarization now reads

$$S=\frac{\sqrt{\det(Z^{\mathrm t}Z+\mathbb{I}_{3\times3}(y^2+X_0^{\mathrm t}X_0))+\mathcal{R}/y^2}}{y^2+X_0^{\mathrm t}X_0}+i\,\frac{\sqrt{\det(ZZ^{\mathrm t}+X_0X_0^{\mathrm t})}}{y(y^2+X_0^{\mathrm t}X_0)} \tag{5.5}$$

where $\mathcal{R}$ stands for

$$\begin{aligned}\mathcal{R}=&\,(y^2+X_0^{\mathrm t}X_0)[(y^2+X_0^{\mathrm t}X_0+\operatorname{tr}ZZ^{\mathrm t})\,X_0^{\mathrm t}ZZ^{\mathrm t}X_0-X_0^{\mathrm t}ZZ^{\mathrm t}ZZ^{\mathrm t}X_0]+\\ &+X_0^{\mathrm t}X_0\,\det Z^{\mathrm t}Z-\det(ZZ^{\mathrm t}+X_0X_0^{\mathrm t})\,.\end{aligned} \tag{5.6}$$

This action generalizes its E_6 counterpart (3.13). Again it is an Born-Infeld-like interpolation between Nambu-Goto and Polyakov formulations. Over and above the winding quantum numbers Z, additional fluxes (y, X_0) are necessary for manifest U-duality invariance. Their interpretation as a target space scalar and three-form also carries through to the 4-torus case: the vector X may be interpreted as a 3-form flux on the world-volume, carrying also a 3-form index in target space. It would be most interesting if the integration over the world-volume SL(3) shape moduli could be carried out for this theory also.

5.2 Membrane/five-brane duality and pure spinors

Going back to the T^3, E_6 case, and in line with the idea of membrane/five-brane duality, it is now interesting to ask if a change of polarization might bring the minimal E_6 theta series to a form which could be interpreted as a five-brane partition function.[10]

Indeed, as noticed in [9], it is possible to choose a polarization where SL(5) becomes linearly realized, instead of SL(3) × SL(3). By Fourier transforming the one-row of the 3×3 matrix, the 11 variables (y, x_0, Z_A^M) rearrange themselves into a 5×5 antisymmetric matrix X,

$$X = \begin{pmatrix} 0 & -\partial_{Z_3^3} & \partial_{Z_3^2} & Z_1^1 & Z_2^1 \\ & 0 & -\partial_{Z_3^1} & Z_1^2 & Z_2^2 \\ & & 0 & Z_1^3 & Z_2^3 \\ & a/s & & 0 & x_0 \\ & & & & 0 \end{pmatrix} \tag{5.7}$$

and a singlet y. In fact, these 11 variables can be supplemented by a 5-vector v_i, expressed in terms of the others as $yv^i = \frac{1}{4}\epsilon^{ijklm} X_{ij} X_{kl}$, in such a way that the 16 variables $(y, X_{ij}, \epsilon_{ijklm} v^m)$ transform as a Majorana-Weyl spinor λ of $SO(5,5) \subset E_6$, subject to the "pure spinor" constraint $\lambda \Gamma^\mu \lambda = 0$, for all $\mu = 1 \dots 10$. This is the direct analog for E_6 of the "string inspired" representation of D_4 on 6 variables m^{ij}, subject to the quadratic constraint $\epsilon_{ijkl} m^{ij} m^{kl} = 0$. A completely analogous construction for the E_7 minimal representation holds also, see [16] for details.

In this polarization, the E_6 spherical vector now takes the very simple form,[11]

$$f_{E6} = \mathcal{K}_1\left(\sqrt{\bar{\lambda}\lambda}\right), \qquad \mathcal{K}_t(x) \equiv x^{-t} K_t(x)\,. \tag{5.8}$$

We now claim that this can be interpreted as the partition function for the five-brane. Indeed, we can view the 5×5 antisymmetric matrix X as the electric flux on the five-brane, $X_{ij} = H_{0ij}$, equal by the self-duality condition to the magnetic components, $X^{ij} = (1/6)\epsilon^{ijklm} H_{klm}$. An SL(6)-invariant partition function for fluxes on a single five-brane has been constructed long ago by Dolan and Nappi in [38], where the SL(6) invariance was non-linearly realized thanks to gaussian Poisson resummation — in other words, the SL(6) symmetry was embedded in an overarching symplectic group Sp(10) acting linearly on the 10 variables X_{ij}. Here, we find that by adding an extra variable y, best thought of as the number of five-branes, the geometric symmetry SL(6) is enlarged to a complete $E_{6(6)}$ group. This is the symmetry group expected for a five-brane wrapped on a T^6 with metric g_{ij}, 3-form C_{ijk} and dual 6-form E_{ijklmn} parameterized by the symmetric space $E_{6(6)}/Usp(4)$. It would be very interesting to use this information to compute the partition function for y overlapping five-branes.

Independently of these mem- and five-brane considerations, let us note that pure spinors of D_5 (albeit with a different real form, SO(1,9)) arise naturally in the description of 10-dimensional Yang-Mills theory: as observed by Berkovits [39], the $N = 1$ superalgebra

[10] We are grateful to L. Motl for forcing us to address this question.

[11] Minimal representations and their spherical vectors have been studied before in this presentation in [37].

in ten dimensions

$$\{Q_\alpha, Q_\beta\} = \Gamma^\mu_{\alpha\beta} P_\mu \,, \tag{5.9}$$

becomes cohomological when auxiliary, pure spinor variables λ^α obeying

$$\lambda \Gamma^\mu \lambda = 0 \,, \tag{5.10}$$

are introduced, because one can then form the nilpotent operator

$$\mathcal{Q} \equiv \lambda^\alpha Q_\alpha \,. \tag{5.11}$$

Superfields are then also functions of pure spinors λ^α. Here, we have shown that the space of functions of λ^α is precisely the minimal representation of E_6. A tantalizing possibility, therefore, is a hidden E_6 symmetry of $d = 10$ super Yang-Mills theory.

6. Discussion

In this Article, we have attempted to compute the R^4 coupling in M-theory compactified on T^3, from a one-loop computation in the microscopic membrane theory. Our basic assumption has been that, at least at the zero-mode level, the membrane theory enjoys an "overarching" symmetry under the minimal representation $E_{6(6)}(\mathbb{Z})$, which mixes world-volume and target space. This provides the most economical way to realize both the modular symmetry $\mathrm{SL}(3,\mathbb{Z})$ on the world-volume, and U-duality $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ in target space. An important consequence is that there should be two discrete degrees of freedom, y and x_0, over and above the nine evident winding numbers $Z^M_A = \partial_A X^M$. After integrating over the worldvolume moduli, only the U-duality symmetry remains.

Using explicit computations and representation theoretic arguments, we have found that the $\mathrm{rank}(Z) = 1$ and $\mathrm{rank}(Z) = 2$ contributions were equal, and reproduced the SL(3) part of the R^4 coupling in (2.5), while the rank 0 and 3 contributions ought reproduce the SL(2) part, i.e., the membrane instanton sum. A fully satisfying proof would require an understanding of the regularization of the poles of the Eisenstein series at $(\lambda_{21}, \lambda_{32}) = (1, 1)$, which we have not been able to obtain thus far. A possible avenue to this end would be to derive the exact expression for the integrated spherical vector beyond the steepest descent approximation (4.25), and study the limit $(y, x_0) \to (0, 0)$. This would shed much desired light on the interpretation of the new quantum numbers (y, x_0), which has remained rather elusive.

On this subject, we may offer the following observations: the E_6 and E_7 cases suggest that the singlet x_0 (resp. the 4-vector X) should correspond to a flux carrying 3-form index both on the membrane world volume. On the other hand, rescaling the winding variables $Z \to yZ$ and also $x_0 \to yx_0$, we observe that the variable y appears linearly as an overall factor in the action (3.13). Therefore, it is natural to identify it with the number of membranes. It would be interesting to understand how to introduce these degrees of freedom in an off-shell membrane action, which would realize the overarching symmetry non-linearly. Equation (3.13) does at least predict the shape of the action of such a theory, restricted to zero-modes. The putative appearance of a membrane number, even calls for a second quantized membrane field theory.

Another interesting question is how does the double dimensional reduction of a membrane winding sum to the T-duality invariant string amplitude work? At the level of the final amplitude (2.5), this is no mystery since this is essentially how it was conjectured in the first place [12]. However, the string amplitude is based on a T-duality/world-volume modular group, dual pair construction for the symplectic groups. An appealing, but *not* necessary, scheme would be to embed these in the exceptional groups proposed here. Unfortunately this seems not to work, since a quick perusal of the branching rules for the E_6 does not yield the requisite Sp(8) subgroup.

Finally, we have presented two observations that reach beyond the subject of supermembranes: first, that the minimal representation of E_6 can also be used to describe stacks of N five-branes wrapped on T^6, in a manifestly U-duality invariant way: it would be very interesting to use this result to compute five-brane partition functions at finite N. Second, pure spinors in ten dimensions carry an action of E_6: the implications for 10-dimensional Yang-Mills theory remain to be uncovered.

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A. Summation measure and degenerate contributions

It is well known that the summation measure of a theta series may be expressed adelically, i.e., as a product over primes of the spherical vector over the p-adic base field (see e.g. [16] for a physicist's review)

$$\mu(y, x_0, Z) = \prod_{p\,\mathrm{prime}} f_{\mathbb{Q}_p}(y, x_0, Z)\,. \tag{A.1}$$

The function f_p is determined by requiring invariance under the maximal compact subgroup $Usp(4, \mathbb{Q}_p)$. Its explicit form can be obtained by noting that the exponent or "action" S in (3.21) has a simple rewriting in terms of the euclidean norm on a certain lagrangian submanifold. Some details: Consider the space of coordinates $X = (x_0, y, Z)$ along with canonical momenta P. Introduce a "hamiltonian"

$$H = \frac{\det Z}{||(x_0, y_0)||^2}\,. \tag{A.2}$$

For now $||v|| \equiv \sqrt{v.v}$ denotes the euclidean norm of a vector v. The lagrangian submanifold is described by the subspace $(X, P = \nabla_X H)$. The action then has a very simple rewriting

$$S = ||(X, \nabla_X H)|| + i\frac{x_0}{y} H\,. \tag{A.3}$$

Recalling that the exponential form of the E_6 spherical vector arose because of the identity $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x)$ we may write

$$f_{\mathbb{R}}^{E_6} = \frac{\mathcal{K}_{1/2}(\mathrm{Re}(S))e^{-i\,\mathrm{Im}(S)}}{||(y,x_0)||^2}\,, \tag{A.4}$$

where $\mathcal{K}_t(x) \equiv x^{-t}K_t(x)$.

The p-adic spherical vector was obtained recently in [21], and takes a very similar form. In the E_6 case, it reads, for $||y||_p \le ||x_0||_p$,

$$f_{\mathbb{Q}_p}^{E_6} = \exp\left(-i\frac{\det Z}{yx_0}\right)||x_0||_p^{-2}\frac{q^2||(X,\nabla_X H_p)||_p^{-2}-1}{q^2-1} \tag{A.5}$$

where now $H_p = \det Z/x_0$ if $||(X,\nabla_X H_p)||_p \le 1$, and 0 otherwise. The opposite case $||y||_p < ||x_0||_p$ is simply obtained by a Weyl reflection, the effect of which is to eliminate the phase and exchange y and x_0. Notice that the p-adic spherical vector can be simply obtained from the real one by replacing the orthogonal norm $||.||^2$ by the p-adic norm and the function $\mathcal{K}_{1/2}$ by its simple (algebraic) p-adic analog. The summation measure is the product of p-adic spherical vectors over all prime p, therefore has support where $||(X,\nabla_X H_p)|| \le 1$ for all p, or equivalently when all entries in X and $\nabla_X H_p$ are natural integers. In particular, if x_0 divides y, then x_0^2 should divide $\det Z$.

In the same Article [21] degenerate terms for exceptional theta series are determined. These terms are familiar to physicists in the context of the large volume expansion of the well known non-holomorphic SL(2) Eisenstein series, see for example equation (2.10). The explicit expression for the E_6 theta series given in (3.21) is the analog of the bulk membrane instanton term in (2.10). In [21], the E_6 degenerate terms are found to be

$$\theta_{E_6} = \sum_{\substack{y\in\mathbb{Z}\backslash\{0\}\\(x_0,Z)\in\mathbb{Z}^{10}}}\left[\prod_{p\ \mathrm{prime}} f_{\mathbb{Q}_p}^{E_6}(y,x_0,Z)\right] f_{\mathbb{R}}^{E_6}(y,x_0,Z) + \sum_{(x_0,Z)}\mu(x_0,Z)\overline{f}(x_0,Z) + \alpha_1 + \alpha_2\,. \tag{A.6}$$

The leading term is the one above in (3.21). As exploited in a cosmological context in [40], the minimal representation has a natural conformal symmetry, the underlying geometry of the eleven dimensional space with variables (y,x_0,X) being conical. The second term in (A.6) corresponds to the limit $y\to 0$ at the tip of the cone: the regulated spherical vector $\overline{f}$ is obtained by dropping the phase factor in (A.4), and setting $y=0$ in the remaining part. The final two terms α_1 and α_2 are automorphic forms built from singlet and minimal representations of the Levi subgroup $\mathrm{SL}(6)\subset E_6$. Again, for complete details, we refer the reader to [21].

B. SL(3, $\mathbb{Z}$) automorphic forms

A most useful account of SL(3, $\mathbb{Z}$) automorphic forms may be found in [32]. In particular, the constant term computations presented in section B.4 are derived in detail there. It was realized long ago, especially by Langlands, that representation theory plays a central rôle in the theory of automorphic forms. Therefore in section B.1 we construct the continuous series irreducible representations of SL(3) upon which $\mathrm{SL}(3,\mathbb{Z})$ invariant automorphic forms are based. We review the fundamental domain construction for $\mathrm{SL}(3,\mathbb{Z})$ in section B.2 (a computation first tackled many years ago by Minkowski!). The remaining sections deal with SL(3) Eisenstein series and their constant terms.

B.1 Continuous representations by unitary induction

All continuous unitary irreducible representations of SL(3) have been constructed long ago in [36], by unitary induction techniques. According to the general philosophy of Kirillov [41], they are in one-to-one correspondence with coadjoint orbits. We recall these results here for convenience.

SL(3) admits a parabolic subgroup P of upper triangular matrices

$$P \equiv \left\{ p = \begin{pmatrix} t_3 & * & * \\ 0 & t_2 & * \\ 0 & 0 & t_1 \end{pmatrix} \;:\; t_1 t_2 t_3 = 1 \right\} \subset \mathrm{SL}(3)\,. \tag{B.1}$$

A one-dimensional representation of P is given by the character

$$\chi(p,\lambda) = \prod_{i=1}^{3} \mathrm{sgn}^{\epsilon_i}(t_i)\, |t_i|^{\rho_i}\,. \tag{B.2}$$

Here $\epsilon_i = \pm 1$ and the three complex variables ρ_i are conveniently parameterized as

$$\rho_1 = \lambda_1 + 1\,, \qquad \rho_2 = \lambda_2\,, \qquad \rho_3 = \lambda_3 - 1\,, \tag{B.3}$$

where the constants represent a shift by the longest root. The unit determinant property implies that all results are written in terms of differences

$$\lambda_{ij} \equiv \lambda_i - \lambda_j\,. \tag{B.4}$$

We will often use the notation $\lambda = (\lambda_{32}, \lambda_{21})$. Functions on SL(3) which transform by a factor $\chi(p,\lambda)$ under left multiplication by $p_0 \in P$ yield a continuous irreducible representation of SL(3). Functions with the above covariance reduce to functions of the three coordinates (x,v,w) of the coset $P\backslash\,\mathrm{SL}(3)$ parameterized by the gauge choice

$$g = \begin{pmatrix} 1 & & \\ x & 1 & \\ v + xw & w & 1 \end{pmatrix}, \tag{B.5}$$

suitably extended to SL(3) via the character χ. The action of the infinitesimal generators of SL(3) on functions $f(x,v,w)$ is computed straightforwardly, and reads

$$\begin{aligned}
E_\gamma &= -\partial_x + w\partial_v\,, \qquad E_{-\gamma} = -x^2\partial_x - v\partial_w + (\lambda_{32}-1)x\,,\\
E_\beta &= \partial_w\,, \qquad E_{-\beta} = w^2\partial_w + vw\partial_v - (v+xw)\partial_x + (1-\lambda_{21})w\,,\\
E_\omega &= \partial_v\,, \qquad E_{-\omega} = v^2\partial_v + vw\partial_w + x(v+xw)\partial_x - (\lambda_{31}-2)v - \lambda_{32}xw\,,
\end{aligned}$$

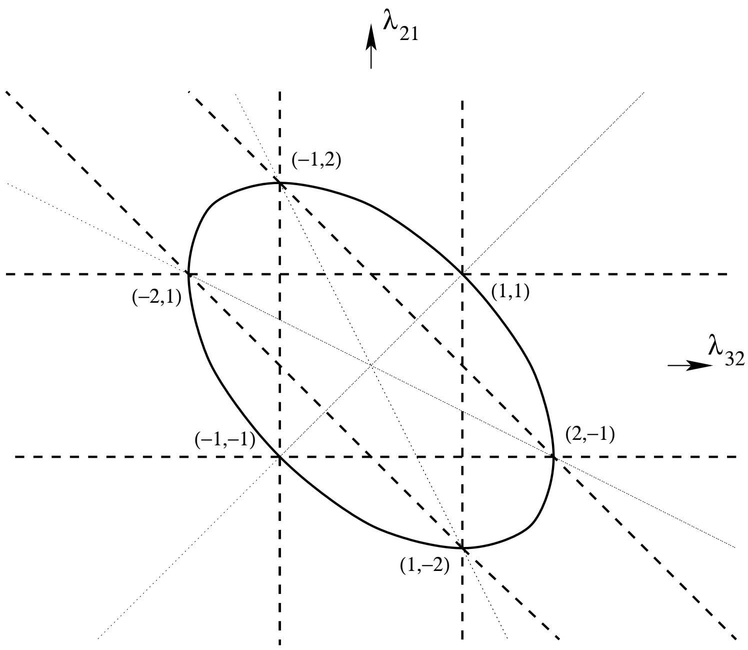

Figure 2: Vanishing loci of $\mathcal{C}_2, \mathcal{C}_3$ in the $(\lambda_{32}, \lambda_{21})$ plane. The quadratic Casimir vanishes on the ellipse, the cubic on the dotted radial lines, while the dashed lines correspond to simple poles of Eisenstein series.

$$H_\gamma = 2x\partial_x + v\partial_v - w\partial_w - (\lambda_{32} - 1)\,, \qquad H_\beta = -x\partial_x + v\partial_v + 2w\partial_w - (\lambda_{21} - 1)\,. \tag{B.6}$$

(Note that the discrete choices of ϵ_i are not visible at the infinitesimal level.) Let us now compute the quadratic and cubic Casimir invariants for this representation as given in (3.23):

$$\mathcal{C}_2 = \frac{1}{6}\left(\lambda_{21}^2 + \lambda_{32}^2 + \lambda_{13}^2\right) - 1\,, \tag{B.7}$$

$$\mathcal{C}_3 = \frac{1}{2}(\lambda_{13} - \lambda_{32})(\lambda_{21} - \lambda_{13})(\lambda_{32} - \lambda_{21})\,. \tag{B.8}$$

We will be particularly interested in the representations such that $\mathcal{C}_2 = \mathcal{C}_3 = 0$. These correspond to $(\lambda_{32}, \lambda_{21}) = (1,1), (2,-1), (1,-2), (-1,-1), (-2,1), (-1,2)$ and appear at the six intersection of the radial lines with the ellipse depicted in figure 2.

The spherical vector in this representation is readily found: The maximal compact subgroup $SO(3) \subset SL(3)$ acts on the columns of the coset representative g in (B.5) as rotations of 3-vectors while the compensating P gauge transformation rescales the first column $(1, x, v + xw)$ by t_3 and the wedge product of first and second rows $(-v, w, 1)$ by $t_3 t_2$. Hence the product of norms raised to the appropriate powers

$$f_{SL(3)} = \left[1 + x^2 + (v + xw)^2\right]^{-s} \; \left[1 + v^2 + w^2\right]^{-t}\,, \tag{B.9}$$

with

$$s \equiv \frac{1}{2}\,\left(1 - \lambda_{32}\right), \qquad t \equiv \frac{1}{2}\,\left(1 - \lambda_{21}\right), \tag{B.10}$$

is simultaneously invariant under the right action of P and left action of SO(3) and is therefore the desired spherical vector.

It is also interesting to obtain the minimal representation from the limit in which the induced representation (B.6) approaches any one of the simple pole dashed lines in figure 2. For example, setting $\lambda_{32} = 1$ and $\lambda_{21} = 1 - 2t$, the induced representation acts faithfully on functions $\phi(v, w)$, by setting $\partial_x = 0$. We then find

$$\begin{aligned} E_\gamma &= w\partial_v \,, & E_{-\gamma} &= -v\partial_w \,, \\ E_\beta &= \partial_w \,, & E_{-\beta} &= w^2\partial_w + vw\partial_v + 2tw \,, \\ E_\omega &= \partial_v \,, & E_{-\omega} &= v^2\partial_v + vw\partial_w - 2tv \,, \\ H_\gamma &= v\partial_v - w\partial_w \,, & H_\beta &= v\partial_v + 2w\partial_w + 2t \,. \end{aligned} \tag{B.11}$$

The spherical vector of this representation is simply $f_{\mathrm{SL}(3),\mathrm{min}} = [1 + v^2 + w^2]^{-t}$ corresponding to an Eisenstein series $E^{\mathrm{SL}(3)}_{\mathbf{3},t}$ in the fundamental representation. Another useful presentation of the minimal representation, used in the Text, is obtained by Fourier transform over $(v, w) \to \frac{1}{i}(\partial_v, \partial_w)$. In this case we find

$$\begin{aligned} E_\gamma &= -v\partial_w \,, & E_{-\gamma} &= w\partial_v \,, \\ E_\beta &= -iw \,, & E_{-\beta} &= iw\partial_w^2 + 3i\partial_w + iv\partial_w\partial_v - 2it\partial_w \,, \\ E_\omega &= -iv \,, & E_{-\omega} &= iv\partial_v^2 + 3i\partial_v + iw\partial_v\partial_w - 2it\partial_v \,, \\ H_\gamma &= -v\partial_v + w\partial_w \,, & H_\beta &= -v\partial_v - 2w\partial_w + 2t - 3 \,. \end{aligned} \tag{B.12}$$

The spherical vector is then

$$\widetilde{f}_{\mathrm{SL}(3),\mathrm{min}} = \mathcal{K}_{1-t}\Big(\sqrt{v^2 + w^2}\Big) \,, \qquad \mathcal{K}_t(x) \equiv x^{-t} K_t(x) \,. \tag{B.13}$$

This form corresponds to the non-degenerate terms in a large volume expansion of the Eisenstein series in the fundamental representation [9].

B.2 Fundamental domain of $\mathrm{SL}(3, \mathbb{Z})$

The fundamental domain of $\mathrm{SL}(3, \mathbb{Z})$ in $\mathrm{SO}(3)\backslash \mathrm{SL}(3, \mathbb{R})$ was first computed by Minkowski [33], here we follow the modern treatment of [34]. We parameterize the coset $\mathrm{SO}(3)\backslash\mathrm{SL}(3, \mathbb{R})$ in the Iwasawa gauge as,

$$e = \begin{pmatrix} \frac{1}{L} & & \\ & \sqrt{\frac{L}{T_2}} & \\ & & \sqrt{LT_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & A_1 & A_2 \\ & 1 & T_1 \\ & & 1 \end{pmatrix} \equiv \frac{1}{L} \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \tag{B.14}$$

where we defined the three vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,, \qquad \vec{e}_2 = \begin{pmatrix} A_1 \\ \sqrt{\frac{L^3}{T_2}} \\ 0 \end{pmatrix} \,, \qquad \vec{e}_3 = \begin{pmatrix} A_2 \\ \sqrt{\frac{L^3}{T_2}}\, T_1 \\ \sqrt{L^3 T_2} \end{pmatrix} \,. \tag{B.15}$$

The fundamental domain is constructed by maximizing a height function

$$h(e) \equiv (\vec{e}_1)_1 \; (\vec{e}_2)_2 \; (\vec{e}_3)_3 = L^3 \,. \tag{B.16}$$

The function h is invariant under the right action of the following $\mathrm{SL}(3,\mathbb{Z})$ elements[12]

$$\mathrm{S}_2=\begin{pmatrix}1&&\\&&1\\&-1&\end{pmatrix},\qquad \mathrm{T}_1=\begin{pmatrix}1&1&\\&1&\\&&1\end{pmatrix},\qquad \mathrm{T}_2=\begin{pmatrix}1&&1\\&1&\\&&1\end{pmatrix},\qquad \mathrm{T}_3=\begin{pmatrix}1&&\\&1&1\\&&1\end{pmatrix},$$
$$\mathrm{U}_1=\begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix},\qquad \mathrm{U}_2=\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix}, \tag{B.18}$$

Therefore, we can always move to a fundamental domain of the $\mathrm{GL}(2,\mathbb{Z})$ in the bottom right hand 2×2 block along with strip conditions on the $A_{1,2}$,

$$T_1^2+T_2^2\geq 1\,,\qquad 0\leq A_1,T_1\leq\frac{1}{2}\,,\qquad -\frac{1}{2}\leq A_2\leq\frac{1}{2}\,. \tag{B.19}$$

There are however additional conditions[13] following from the actions of further $\mathrm{SL}(3,\mathbb{Z})$ elements on the height function

$$\begin{aligned}
\mathrm{S}_1&=\begin{pmatrix}&1&\\-1&&\\&&1\end{pmatrix}:h\mapsto\frac{h}{|\vec{e}_2|^3}\\
\mathrm{S}_3&=\begin{pmatrix}&&1\\&1&\\-1&&\end{pmatrix}:h\mapsto\frac{h}{|\vec{e}_3|^3}\\
\mathrm{S}_4&=\begin{pmatrix}&1&\\1&&-1\\-1&&\end{pmatrix}:h\mapsto\frac{h}{|\vec{e}_3-\vec{e}_2|^3}\\
\mathrm{S}_5&=\begin{pmatrix}1&&\\-1&1&\\1&&1\end{pmatrix}:h\mapsto\frac{h}{|\vec{e}_3-\vec{e}_2+\vec{e}_1|^3}\,.
\end{aligned} \tag{B.20}$$

Clearly[14] maximal h requires $|\vec{e}_2|$, $|\vec{e}_3|$, $|\vec{e}_3-\vec{e}_2|$, $|\vec{e}_3-\vec{e}_2+\vec{e}_1|\geq 1$, i.e.,

$$1\leq A_1^2+\frac{L^3}{T_2}$$

[12]Note that the set of elements $\{S_{1,2,3,4,5},T_{1,2,3},U_{1,2}\}$ is convenient for determining the fundamental domain, yet overcomplete-$\mathrm{PSL}(2,\mathbb{Z})$ is generated by the shift and clock matrices

$$\begin{pmatrix}1&&1\\&1&\\&&1\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}&1&\\&&1\\1&&\end{pmatrix}. \tag{B.17}$$

[13]Some of these were missed in [8], although this omission does not alter the conclusions found there.

[14]The proof: Without loss of generality assume $A_1\geq 0$. Then if $|\vec{e}_2|<1$, act with S_1 to increase h. Since $\mathrm{S}_1:|\vec{e}_2|\mapsto 1/|\vec{e}_2|$, further actions of S_1 do not increase h. Next act with S_3 in the case that $|\vec{e}_3|<1$. A single action suffices because $\mathrm{S}_3:|\vec{e}_3|\mapsto 1/|\vec{e}_3|$. Furthermore, since $\mathrm{S}_3:|\vec{e}_2|\mapsto|\vec{e}_2|/|\vec{e}_3|$ this does not disturb the previous $|\vec{e}_2|\geq 1$ inequality. Similarly, if $|\vec{e}_3-\vec{T}|<1$, act with S_4, which does not ruin previous equalities as $\mathrm{S}_4:(|\vec{e}_2|,|\vec{e}_3|)\mapsto(1/|\vec{e}_3-\vec{e}_2|,|\vec{e}_2|/|\vec{e}_3-\vec{e}_2|)$. Also, only one S_4 action

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$$
\begin{aligned}
1 &\leq A_2^2 + \frac{L^3}{T_2} T_1^2 + L^3 T_2 \\
1 &\leq (A_2 - A_1)^2 + \frac{L^3}{T_2}(T_1 - 1)^2 + L^3 T_2 \\
1 &\leq (A_2 - A_1 + 1)^2 + \frac{L^3}{T_2}(T_1 - 1)^2 + L^3 T_2 \,.
\end{aligned} \tag{B.21}
$$

Conditions (B.19) and (B.21) constitute the fundamental domain since the (non-minimal) set of elements considered generate $\mathrm{SL}(3,\mathbb{Z})$. Finally, note that in these variables, the $\mathrm{SL}(3)_L$ invariant integration measure is

$$
de = \frac{d^2T}{T_2^2}\, d^2A\, \frac{dL}{L^4}\,. \tag{B.22}
$$

B.3 SL(3) Eisenstein series

In this appendix we review prescient aspects of the theory of Eisenstein series and gather formulae needed in the main text for SL(3). For our purposes it suffices to consider Eisenstein series based on the minimal parabolic subgroup

$$
P = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\} \subset \mathrm{SL}(3,\mathbb{R})\,, \tag{B.23}
$$

which we decompose into its nilpotent radical and Levi components

$$
P = MN\,, \qquad M = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}. \tag{B.24}
$$

A general element $g \in \mathrm{SL}(3,\mathbb{R})$ has the Iwasawa decomposition

$$
g = k \begin{pmatrix} t_3 & & \\ & t_2 & \\ & & t_1 \end{pmatrix} \begin{pmatrix} 1 & a_3 & a_2 \\ & 1 & a_1 \\ & & 1 \end{pmatrix} \tag{B.25}
$$

where $t_1 t_2 t_3 = 1$ and $k \in \mathrm{SO}(3)$ the maximal compact subgroup of $\mathrm{SL}(3,\mathbb{R})$. We can then introduce the function on SL(3)

$$
F(g;\lambda) \equiv t_1^{\lambda_1+1} t_2^{\lambda_2} t_3^{\lambda_3-1} = t_2^{\lambda_{21}-1} t_3^{\lambda_{31}-2}\,, \qquad \lambda_{ij} \equiv \lambda_i - \lambda_j\,, \tag{B.26}
$$

which is manifestly invariant under left action of the compact SO(3) and right action of the nilpotent N. It is also an eigenfunction of the $\mathrm{SL}(3,\mathbb{R})$-invariant differential operators $\Delta^{(2,3)}_{\mathrm{SL}(3)}$ introduced in section 3.6.

is needed because $\mathrm{S}_4 : |\vec{e}_3 - \vec{e}_2| \mapsto |\vec{e}_1 + \vec{e}_2|/|\vec{e}_3 - \vec{e}_2|$ and $|\vec{e}_1 + \vec{e}_2| \geq 1$ for $A_1 \geq 0$. Finally, when $|\vec{e}_3 - \vec{e}_2 + \vec{e}_1| < 1$, act with S_5. This respects previous inequalities because $\mathrm{S}_5 : (|\vec{e}_2|, |\vec{e}_3|, |\vec{e}_3 - \vec{e}_2|) \mapsto (|\vec{e}_2|/|\vec{e}_3 - \vec{e}_2 + \vec{e}_1|, |\vec{e}_3|/|\vec{e}_3 - \vec{e}_2 + \vec{e}_1|, |\vec{e}_3 - \vec{e}_2|/|\vec{e}_3 - \vec{e}_2 + \vec{e}_1|)$. Only finitely many applications of S_5 are necessary because $\mathrm{S}_5 : |n(\vec{e}_3 - \vec{e}_2) + \vec{e}_1| \mapsto |(n+1)(\vec{e}_3 - \vec{e}_2) + \vec{e}_1|/|\vec{e}_3 - \vec{e}_2 + \vec{e}_1|$. **QED**

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To produce an automorphic form, we average (automorphize) $F(g;\lambda)$ over the right action of $\mathrm{SL}(3,\mathbb{Z})$. Indeed since $F(g;\lambda) = F(gn;\lambda)$ for $n \in N$ it suffices to sum over $N(\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{Z})$ (where $N(\mathbb{Z}) \equiv \mathrm{SL}(3,\mathbb{Z}) \cap P$). Therefore, the SL(3) Eisenstein series for the minimal parabolic is defined as

$$E(g;\lambda) = \sum_{\gamma \in N(\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{Z})} F(g\gamma;\lambda)\,. \tag{B.27}$$

This is now manifestly invariant under left action of SO(3) and right action of $\mathrm{SL}(3,\mathbb{Z})$, therefore an automorphic function on $\mathrm{SO}(3)\backslash \mathrm{SL}(3)$. As usual, this automorphic form can be decomposed as in (3.3), where the representation ρ is precisely the representation induced from the minimal parabolic P with character (B.2).[15]

Let us spell out this formula explicitly in a form perhaps more familiar to physicists: Firstly observe that an integer valued matrix belongs to $\mathrm{SL}(3,\mathbb{Z})$ only if the greatest common divisor of any row (or column) is unity. Let $m^{\mathrm{t}} \equiv (m^1, m^2, m^3)$, $n^{\mathrm{t}} \equiv (n^1, n^2, n^3)$ where $\gcd(m^i) = \gcd(n^i) = 1$. The summation matrix γ is subject to the equivalence relation

$$\begin{aligned}\gamma = \begin{pmatrix} m & n & p \end{pmatrix} &\sim \gamma n = \begin{pmatrix} m & n{+}n_3 m & p{+}n_1\, n{+}n_2 m \end{pmatrix}\\ n \equiv &\begin{pmatrix} 1 & n_3 & n_2 \\ & 1 & n_1 \\ & & 1 \end{pmatrix} \in N(\mathbb{Z})\,.\end{aligned} \tag{B.28}$$

Note that the column vector p^* is completely determined modulo integer shifts by m and n once these two row vectors are specified. Therefore the summation over $\gamma \in N(\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{Z})$ amounts to summing over m^i and n^i subject to $\gcd(m^i) = \gcd(n^i) = 1$ and $n \sim n + n_3 m$, $n_3 \in \mathbb{Z}$.

To display the summand of (B.27) we compute the Iwasawa decomposition of $g\gamma$

$$g\gamma = k' \begin{pmatrix} \sqrt{m^{\mathrm{t}} G m} & * & * \\ & \sqrt{\frac{m^{\mathrm{t}} G m \; n^{\mathrm{t}} G n - (m^{\mathrm{t}} G n)^2}{m^{\mathrm{t}} G m}} & * \\ & & \frac{1}{\sqrt{m^{\mathrm{t}} G m \; n^{\mathrm{t}} G n - (m^{\mathrm{t}} G n)^2}} \end{pmatrix}, \tag{B.29}$$

where $k' \in \mathrm{SO}(3)$ and the matrix $G \equiv g^{\mathrm{t}} g$. Orchestrating, we find

$$E(g;\lambda) = \sum_{\substack{\{(m,n)\in\mathbb{Z}^3\otimes\mathbb{Z}^3:\\ \gcd(m)=\gcd(n)=1,\\ m\neq 0,\; n\sim n+zm\neq 0\}}} \Big(m^{\mathrm{t}} G m\Big)^{-s} \Big(m^{\mathrm{t}} G m \; n^{\mathrm{t}} G n - (m^{\mathrm{t}} G n)^2\Big)^{-t}$$

---

[15] Indeed, the right action on g can be converted into a left action on γ, which takes value in $P\backslash \mathrm{SL}(3)$. Eisenstein series for the minimal parabolics are defined in a similar way. For example, for P_1 defined in (B.34), elements of $\mathrm{SL}(3,\mathbb{R})$ are decomposed as

$$g = k \begin{pmatrix} t_3 & & \\ & \frac{1}{\sqrt{t_3}} & \\ & & \frac{1}{\sqrt{t_3}} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{\tau_2}} & \\ & & \sqrt{\tau_2} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \tau_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 & a_2 \\ & 1 & \\ & & 1 \end{pmatrix}, \qquad k \in \mathrm{SO}(3)\,.$$

Then instead of automorphizing $F(g;\lambda)$, one averages over $\phi(\tau_1 + i\tau_2)\, t_3^{\lambda_3 - 1}$ where ϕ is an SL(2) cusp form.

$$= \frac{1}{4\zeta(2s+2t)\zeta(2t)} \sum_{\substack{\{(m,n)\in\mathbb{Z}^3\otimes\mathbb{Z}^3:\\ m\neq 0,\ n\sim n+zm\neq 0\}}} \left(m^{\mathrm{t}}Gm\right)^{-s}\left(m^{\mathrm{t}}Gm\ n^{\mathrm{t}}Gn - (m^{\mathrm{t}}Gn)^2\right)^{-t}, \tag{B.30}$$

where

$$2s \equiv 1-\lambda_{32}\,, \qquad 2t \equiv 1-\lambda_{21}\,. \tag{B.31}$$

In particular, observe that, in accordance with (B.5), if we set $m^{\mathrm{t}} = (1, x, v+xw)$ and $n^{\mathrm{t}} = (0,1,w)$ (which makes sense adelically) the summand in (B.30) matches the spherical vector (B.9).

B.4 Constant term computations

By construction, the Eisenstein series $E(g;\lambda)$ is invariant under right action of $\mathrm{SL}(3,\mathbb{Z})$ and therefore of its Borel subgroup $N(\mathbb{Z})$, which acts on (a_1,a_2,a_3) as

$$E(a_1,a_2,a_3) = E(a_1+n_1, a_2+n_2+n_1a_3, a_3+n_3)\,, \qquad n_i\in\mathbb{Z}\,. \tag{B.32}$$

(we drop the dependence on t_i and λ_i). This Heisenberg group includes two distinct $\mathbb{Z}\times\mathbb{Z}$ subgroups,

$$N_1 \equiv \left\{\begin{pmatrix}1 & * & * \\ & 1 & \\ & & 1\end{pmatrix}\right\}, \qquad N_2 \equiv \left\{\begin{pmatrix}1 & & * \\ & 1 & * \\ & & 1\end{pmatrix}\right\}, \tag{B.33}$$

which can be viewed as the unipotent radicals of two associate maximal parabolic subgroups of $\mathrm{SL}(3,\mathbb{Z})$,

$$P_1 \equiv \left\{\begin{pmatrix}* & * & * \\ & * & * \\ & * & *\end{pmatrix}\right\} \subset \mathrm{SL}(3,\mathbb{R}) \supset \left\{\begin{pmatrix}* & * & * \\ * & * & * \\ & & *\end{pmatrix}\right\} \equiv P_2\,. \tag{B.34}$$

One may therefore construct the two Fourier series expansions

$$E(a_1,a_2,a_3) = \sum_{m_2,m_3} e^{2\pi i(m_2a_2+m_3a_3)} E_{P_1}^{m_2,m_3}(a_1) \tag{B.35}$$

$$= \sum_{m_1,m_2} e^{2\pi i(m_1a_1+m_2[a_2-a_1a_3])} E_{P_2}^{m_1,m_2}(a_3)\,. \tag{B.36}$$

The Fourier zero-modes $E_{P_1}^{0,0}(a_1) \equiv E_{P_1}(g,\lambda)$ and $E_{P_2}^{0,0}(a_3) \equiv E_{P_3}(g,\lambda)$, are called the constant terms of the Eisenstein series $E(g;\lambda)$ with respect to parabolics P_1, P_2, respectively. Evidently they may also be obtained by performing compact integrations

$$E_{P_i}(g,\lambda) = \int_{N_i/[\mathrm{SL}(3,\mathbb{Z})\cap N_i]} dn\ E(gn;\lambda)\,. \tag{B.37}$$

This is a general definition for constant terms. A third constant term computation, relative to the minimal parabolic, is also possible

$$E_P(g,\lambda) = \int_{N/[\mathrm{SL}(3,\mathbb{Z})\cap N]} dn\ E(gn;\lambda)\,. \tag{B.38}$$

The $N/[\mathrm{SL}(3,\mathbb{Z})\cap N]$ is simply the set of upper triangular matrices

$$N/[\mathrm{SL}(3,\mathbb{Z})\cap N] = \left\{ \begin{pmatrix} 1 & n_1 & n_2 \\ & 1 & n_3 \\ & & 1 \end{pmatrix} : 0 \le n_1, n_2, n_3 < 1 \right\}. \tag{B.39}$$

Hence, from the Fourier expansions (B.35) and (B.36) we find

$$E_P(g;\lambda) = \int_0^1 dn_1 E^{0,0}_{P_1}(a_1+n_1) = \int_0^1 dn_3 E^{0,0}_{P_2}(a_3+n_3)\,. \tag{B.40}$$

These relations provide a useful check on the constant term computations.

Although constant term computations amount to the usual large volume expansion in String Theory-amenable to explicit computations via Schwinger's integral representation and Poisson resummation, for general parabolics and higher groups this method becomes unwieldy. Fortunately the task of computing SL(3) constant terms has already been tackled by mathematicians. Here we rely heavily on the analysis of [32].

First, recall the SL(2) case

$$\begin{aligned} \sum_{\substack{(m,n)\neq 0,\\ \gcd(m,n)=1}} \left(\frac{\tau_2}{|m+n\tau|}\right)^s &= \frac{1}{2\zeta(s)} \sum_{(m,n)\neq 0} \left(\frac{\tau_2}{|m+n\tau|}\right)^s \\ &= \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\,\tau_2^{1-s} + \sum_{n\neq 0} a_n(\tau_2)\, e^{2\pi i n\tau_1}\,. \\ &\equiv E_{1-2s}(\tau)\,. \end{aligned} \tag{B.41}$$

Here

$$\xi(s) \equiv \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\,, \tag{B.42}$$

is the completed Riemann Zeta function and

$$a_n(\tau_2) = \frac{\sqrt{\tau_2}}{\xi(2s)}\,\mu_s(n) n^{s-1/2} K_{s-1/2}(2\pi\tau_2|n|)\,, \qquad \mu_s(n) \equiv \sum_{m|n} m^{-2s+1}\,. \tag{B.43}$$

Note that, in particular,

$$\int_0^1 d\tau_1 E_s(\tau) = \tau_2^{\frac{1}{2}-\frac{1}{s}} + \frac{\xi(-s)}{\xi(s)}\,\tau_2^{\frac{1}{2}+\frac{1}{s}}\,. \tag{B.44}$$

Here $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann Zeta function. Selberg's functional relation

$$\xi(s)E_s(\tau) = \xi(-s)E_{-s}(\tau)\,, \tag{B.45}$$

may be obtained from the constant terms. Indeed, multiplying (B.41) by $\zeta(2s)$ and using $\xi(s) = \xi(1-s)$ we see that these are indeed invariant under $s \to 1-s$. Note that the symmetric form of the SL(2) Eisenstein series in (B.45) is related to the usual one based on the fundamental representation of SL(2) by $E_{1-2s}(\tau) = 2\zeta(2s)E^{\mathrm{SL}(2)}_{\mathbf{2},s}$.

For SL(3) the following formulae hold:

$$E_{P_1}(g,\lambda) = t_3^{\lambda_{31}-\frac{1}{2}\lambda_{21}-\frac{3}{2}} E_{\lambda_{21}}(\tau) + \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} t_3^{\lambda_{21}-\frac{1}{2}\lambda_{31}-\frac{3}{2}} E_{\lambda_{31}}(\tau) + \\ + \frac{\xi(\lambda_{12})\xi(\lambda_{13})}{\xi(\lambda_{21})\xi(\lambda_{31})} t_3^{\lambda_{12}-\frac{1}{2}\lambda_{32}-\frac{3}{2}} E_{\lambda_{32}}(\tau)\,, \tag{B.46}$$

where $\tau = a_1 + i\frac{t_1}{t_2}$ in this formula;

$$E_{P_2}(g,\lambda) = t_1^{\lambda_{12}-\frac{1}{2}\lambda_{32}+\frac{3}{2}} E_{\lambda_{32}}(\tau) + \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} t_1^{\lambda_{21}-\frac{1}{2}\lambda_{31}+\frac{3}{2}} E_{\lambda_{31}}(\tau) + \\ + \frac{\xi(\lambda_{23})\xi(\lambda_{13})}{\xi(\lambda_{32})\xi(\lambda_{31})} t_1^{\lambda_{31}-\frac{1}{2}\lambda_{21}+\frac{3}{2}} E_{\lambda_{21}}(\tau)\,, \tag{B.47}$$

here $\tau = a_3 + i\frac{t_2}{t_3}$.

By virtue of (B.40), inserting the constant term result for the SL(2) Eisenstein series (B.44) in either (B.35) or (B.36), yields

$$\begin{aligned} E_P(g,\lambda) &= t_1^{\lambda_1+1} t_2^{\lambda_2} t_3^{\lambda_3-1} + t_1^{\lambda_2+1} t_2^{\lambda_1} t_3^{\lambda_3-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} + \\ &\quad + t_1^{\lambda_1+1} t_2^{\lambda_3} t_3^{\lambda_2-1} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} + t_1^{\lambda_3+1} t_2^{\lambda_1} t_3^{\lambda_2-1} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} + \\ &\quad + t_1^{\lambda_2+1} t_2^{\lambda_3} t_3^{\lambda_1-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} + t_1^{\lambda_3+1} t_2^{\lambda_2} t_3^{\lambda_1-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} \\ &= \sum_{s\in S(3)} t_1^{\lambda_{s(1)}+1} t_2^{\lambda_{s(2)}} t_3^{\lambda_{s(3)}-1} \prod_{\substack{1\le i<j\le 3 \\ s^{-1}(i)>s^{-1}(j)}} \frac{\xi(\lambda_{ij})}{\xi(\lambda_{ji})}\,. \end{aligned} \tag{B.48}$$

In particular observe that

$$\xi(\lambda_{21})\xi(\lambda_{31})\xi(\lambda_{32}) E_P(g,\lambda) = \sum_{s\in S(3)} t_1^{\lambda_{s(1)}+1} t_2^{\lambda_{s(2)}} t_3^{\lambda_{s(3)}-1} \xi(\lambda_{s(2)s(1)})\xi(\lambda_{s(3)s(1)})\xi(\lambda_{s(3)s(2)})\,, \tag{B.49}$$

is manifestly symmetric under the action of the Weyl group $S(3)$ on λ. This relation in fact extends to the Eisenstein series $E(g,\lambda)$ and is the generalization of the Selberg functional relation (B.45) to SL(3).

Another interesting calculation is to compute the constant terms in the case $2t = 1-\lambda_{21} = 0$, i.e. when only the quadratic summand is present in (B.30) which ought correspond to the fundamental representation Eisenstein series $E^{\mathrm{SL}(3)}_{\mathbf{3},s}(g)$. Setting $t=0$ in (B.30) leads to a divergent sum over n but the analytic continuation to $t=0$ is regular and can be obtained by studying the constant term formulae:

$$\begin{aligned} E_{P_1}(g;1-2s,1) &= t_3^{-2s} + t_3^{s-3/2} \frac{\xi(2s-1)}{\xi(2s)} E_{2-2s}(\tau) \\ E_{P_2}(g;1-2s,1) &= t_1^{-2s+3} \frac{\xi(2s-2)}{\xi(2s)} + t_1^{s} E_{1-2s}(\tau)\,. \end{aligned} \tag{B.50}$$

Here we used $E_1(\tau) = 1$ and the fact that $\xi(s)$ has a pole at $s = 1$. Indeed these results agree with those for the SL$(d+1)$ Eisenstein series in the fundamental representation defined by

$$\overline{E}^{\mathrm{SL}(d+1)}_{\mathbf{d+1},s}(g) \equiv \sum_{\substack{m\in\mathbb{Z}^{d+1}\backslash\{0\}\\ \gcd(m)=1}} \left(\frac{1}{mGm^{\mathrm{t}}}\right)^s \equiv \frac{1}{2\zeta(2s)} E^{\mathrm{SL}(d+1)}_{\mathbf{d+1},s}(g)\,, \tag{B.51}$$

for which a simple large volume computation yields

$$\int_0^1 d^d n \overline{E}^{\mathrm{SL}(d+1)}_{\mathbf{d+1},s}(gn) = \begin{cases} t_{d+1}^{-2s} + \dfrac{\xi(2s-1)}{\xi(2s)} t_{d+1}^{\frac{2s-1}{d}-1} \overline{E}^{\mathrm{SL}(d)}_{\mathbf{d},s-1/2}(\gamma)\,,\ n = \begin{pmatrix} 1 & n_1 & \cdots & n_d \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \\ \\ t_1^{1+d-2s}\dfrac{\xi(2s-d)}{\xi(2s)} + t_1^{2s/d}\overline{E}^{\mathrm{SL}(d)}_{\mathbf{d},s}(\gamma)\,, \quad n = \begin{pmatrix} 1 & & & n_1 \\ & 1 & & n_2 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}. \end{cases}$$

Note that $\gamma \in \mathrm{SL}(d)$ is obtained by respectively decomposing $g = a\gamma n$ with $n \in N_{1,d}$ and $a \in M_{1,d}$. In the case $d = 2$, $\overline{E}^{\mathrm{SL}(2)}_{\mathbf{2},s}(\gamma) = E_{1-2s}(\tau)$ with τ as in footnote 15.

C. Membrane world-volume integrals

An alternative proposal to obtain an SL(2) automorphic form from an SL(3) one, is to integrate over a volume modulus by decomposing $\mathrm{SL}(3)_{NL} \supset \mathrm{SL}(2)_U \times \mathbb{R}^+$. Here we perform this computation for a simple toy model-an SL(3) Eisenstein series in the fundamental representation $E^{\mathrm{SL}(3)}_{\mathbf{3},s}(g_{NL})$. Firstly, we decompose the SL(3) moduli as

$$g_{NL} = \left(\begin{array}{c|c} \nu g_U & 0 \\ \hline 0 & \frac{1}{\nu^2} \end{array}\right). \tag{C.1}$$

Holding the 2×2 metric $g_U = g_U(\tau)$ constant, we can compute the metric on the membrane world-volume modulus $d\sigma^2 = -\frac{1}{6}\,\mathrm{tr}\ dgdg^{-1} = \frac{d\nu^2}{\nu^2}$ to determine the measure of integration. Therefore we study

$$\int_0^\infty \frac{d\nu}{\nu} E^{\mathrm{SL}(3)}_{\mathbf{3},s}(g) = \int_0^\infty \frac{d\nu}{\nu} \sum_{\substack{\{(\mu,m):\mu\in\mathbb{Z}^2,\, m\in\mathbb{Z}\\ (\mu,m)\neq(0,0,0)\}}} \left(\frac{1}{\nu\mu^{\mathrm{t}}\gamma\mu + \frac{m^2}{\nu^2}}\right)^s \tag{C.2}$$

$$= 2\,G(s)\,\zeta\left(\frac{2s}{3}\right) E^{\mathrm{SL}(2)}_{\mathbf{2},\frac{2s}{3}}(g_U) + \mathcal{A}\,, \tag{C.3}$$

where

$$G(s) \equiv \int_0^\infty \frac{d\nu}{\nu^{1-2s}(1+\nu^3)^s} = \frac{\Gamma(\frac{s}{3})\Gamma(\frac{2s}{3})}{3\Gamma(s)}\,. \tag{C.4}$$

The complete integral is however divergent, because of the degenerate contributions at $m = 0$ or $\mu = (0, 0)$, which are given by the formal expression

$$\mathcal{A} = \int_0^\infty \frac{d\nu}{\nu^{1+s}} \left(\frac{1}{2} + E^{\mathrm{SL}(2)}_{\mathbf{2},s}(g_U) \right) . \tag{C.5}$$

This supports the conclusion in the Text that the membrane volume integration is divergent. We have not been able to find a consistent regularization scheme.

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